Large Deviations Technique on Stochastic Reaction-Diffusion Equations

by

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Contents

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Declaration

The author declares that, to her best knowledge that the work contained within this thesis is original and her own work under the supervision of her supervisor, Dr. Roger Tribe. Also, the material in this thesis is submitted for the degree of PhD. to the University of Warwick only and has not been submitted to any other universities.

Abstract

There are two different problems studied in this thesis. The first one is a travelling wave problem. We will improve the result proved in [4] to derive the ergodic property of the travelling wave behind the wavefront. The second problem is a large deviation problem concerning solutions to certain kind stochastic partial differential equations. We will first briefly introduce some basics about SPDE in chapter 2. In chapter 3, we will prove a large deviation principle for super-Brownian motion when it is considered as a solution to an SPDE, using the LDP for super-Brownian motion when it is considered as a measure-valued branching process as solution to a martingale problem. In chapter 4, we will prove another LDP result for solutions of a stochastic reaction-diffusion equation with degenerate noise term. Finally in chapter 5, we will explore some applications of those LDP results proved previously.

Chapter 1

Wave Speed for Stochastic KPP Equations

In this chapter, we will consider the following one dimensional stochastic generalized KPP equation

$$\partial_t u(t,x) \quad = \quad \frac{D}{2}\triangle u(t,x) + u(t,x)c\left(u(t,x)\right) + u(t,x)\dot{W}_t, \qquad (1.1)$$

for $x \in R$ and $t \in [0,\infty)$, with initial condition $u(0,x) = u_0(x) = \mathbf{1}_{(-\infty,0]}(x)$, where D is a positive constant, W is a Brownian motion on a probability space $(\Omega, \mathscr{F}, P)$ and the function c satisfies certain conditions which will be specified later.

First we will briefly review some work concerning travelling wave problems for reaction diffusion equations with and without noise perturbation.

Kolmogorov, Petrovskii and Piskunov [2] and Fisher [30] considered the semilinear reaction diffusion equation

$$\begin{aligned} \partial_t u(t,x) &= \frac{D}{2}\triangle u(t,x) + u(t,x)\hat{c}\left(1 - u(t,x)\right), \qquad (1.2) \\ u(0,x) &= u_0(x) = \mathbf{1}_{(-\infty,0]}(x), \end{aligned}$$

where D and $\hat{c}$ are positive constants. It is easy to check that for each $t > 0$, $u(t,x)$ is a strictly monotone function decreasing from 1 as $x \to -\infty$ to 0 at $x \to \infty$. Then there exists a unique $\theta(t)$ such that $u(t, \theta(t)) = 1/2$. It was proved in [2] that $\lim_{t\to\infty}\frac{1}{t}\theta(t) = \sqrt{2\hat{c}D}$ and that $\lim_{t\to\infty} u(t,\theta(t)+x) = U(x)$, where $U(x)$ is the solution of the ODE

$$\frac{D}{2}U^{''}(x) + \sqrt{2\hat{c}D}U^{'}(x) + U(x)\hat{c}(1-U(x)) = 0,\ -\infty < x < \infty,$$

with conditions $\lim_{x\to\infty} U(x) = 0$, $\lim_{x\to-\infty} U(x) = 1$, and $U(0) = 1/2$. This ODE has a unique solution. Roughly speaking this means that the solution to the original reaction diffusion equation (1.2) behaves for large t as a wave $U(x - t\sqrt{2\hat{c}D})$ and it can be characterized by its shape $U(x)$ and the speed $\sqrt{2\hat{c}D}$. Freidlin ([25], [24], [26], [27]) considered generalized KPP equation where, in the reaction-diffusion (1.2), he considered more general nonlinear term instead of $\hat{c}u(1-u)$. He defined the asymptotic wave speed α to be the number such that: for any $h > 0$, $\lim_{t\to\infty}\sup_{x>(\alpha+h)t} u(t,x) = 0$ and $\lim_{t\to\infty}\inf_{x<(\alpha-h)t} u(t,x) = 1$. He used Feynman-Kac formula and large deviation theory to prove the existence of travelling waves. Zhao and Elworthy [9] considered the scaled equation with small parameter μ

$$\begin{aligned}
\partial_t u^{\mu}(t,x) &= \frac{D}{2}\mu^2 \triangle u^{\mu}(t,x) + \frac{1}{\mu^2}u^{\mu}(t,x)c\left(x,u^{\mu}(t,x)\right),\\
u^{\mu}(0,x) &= T_0(x)\exp\left(-\frac{1}{\mu^2}S_0(x)\right).
\end{aligned}$$

They used Feymann-Kac formula, Maruyama-Girsanov-Cameron-Martin formula and Hamilton-Jacobi theory from classical mechanics to prove the convergence of the solution behind and ahead of the wave front, they also proved the shape of the wave ahead of the wave front. Elworthy, Truman and Zhao [18] considered similar equation but in which the initial condition is as $u^{\mu}(0,x) = \sum_{j=1}^{N} T_{j,0}(x)\exp\left(-\frac{1}{\mu^2}S_{j,0}(x)\right)$, or is a step function. They used Maruyama-Girsanov-Cameron-Martin formula and Varadhan's theorem to obtain the propagation of the wavefront and the shape of the wave ahead of the wavefront.

Stochastic generalized KPP equations like (1.1) were studied in [19], [14] and [3]. In [19] and [14], the authors used Hamilton-Jacobi theory and Freidlin's idea ([24], [25], [26], [27]) to study the scaled equation for small parameter μ

$$\begin{aligned} \partial_t u^\mu(t,x) &= \frac{1}{2}\mu^2 \triangle u^\mu(t,x) + \frac{1}{\mu^2} u^\mu(t,x) c\left(x, u^\mu(t,x)\right) \\ &\quad + u^\mu(t,x) F(\mu, k(t)) \dot{W}_t, \\ u(0,x) &= T_0(x) \exp\left(-\frac{1}{\mu^2} S_0(x)\right), \end{aligned} \tag{1.3}$$

where $F(\mu, k(t)) = k(t)$ (weak noise), $F(\mu, k(t)) = \frac{1}{\mu} k(t)$ (mild noise) and $F(\mu, k(t)) = \frac{1}{\mu^2} k(t)$ (strong noise). In [3], the authors considered the unscaled equation

$$\begin{aligned} \partial_t u(t,x) &= \frac{D}{2} \triangle u(t,x) + u(t,x) c\left(u(t,x)\right) + k(t) u(t,x) \dot{W}_t, \\ u(0,x) &= u_0(x), \end{aligned} \tag{1.4}$$

where $\int_0^\infty k^2(s)ds < \infty$ corresponds to the case of weak noise, the case of mild noise corresponds to the condition $\lim_{t\to\infty} \sqrt{\frac{1}{t}\int_0^t k^2(s)ds} < \sqrt{2c(0)}$, and $\lim_{t\to\infty} \sqrt{\frac{1}{t}\int_0^t k^2(s)ds} > \sqrt{2c(0)}$ corresponds to the case of strong noise. In [3], the authors gave a proper definition of strong solution to the stochastic generalized KPP equation, and the existence of the solution and its regularity (actually contained in the definition given in [3]) was pointed out in [19], [14], and proved in [3]. It was proved that in the case of strong noise, the noise will destroy the wave and force the solution to die; in the case of weak noise, the effect of the noise is so small that the wave would tend to the solution of the corresponding deterministic equation as time tends to infinity (or in the scaled case as μ goes to 0); in the case of mild noise, there will be a residual wave propagating at a different speed to that of the deterministic equation. Also, it was proved in these papers that in the case of mild noise, ahead of the wavefront, the solution is exponentially small as time goes to infinity. But there is not such kind convergence results about the solution behind the wavefront as the solution is oscillatory. As suggested by numerical works (Appendix in [19] and [14]), behind the wavefront, $\frac{1}{t}\int_0^t u(s,x)ds$ might have a simple form. This problem was studied in [4]. The authors first studied the corresponding stochastic ordinary equation and used Feynman-Kac formula and Freidlin's idea to obtain a comparison result of the solution to the original stochastic generalized KPP equation and the solution of the SDE and hence obtained an upper and lower bound for $\frac{1}{t}\int_0^t u(s,x)ds$ behind the wavefront. But in their result, the lower bound for $\frac{1}{t}\int_0^t u(s,x)ds$ is only obtained for the region a bit further behind the wavefront, not for all the region behind the wavefront. In this chapter, we will improve a key estimate and then use essentially the same method to give this lower bound on all the region behind the wavefront.

Before we start, we first state here the conditions on the function c appearing in the stochastic generalized KPP equation (1.1). We suppose the function c satisfies

C1 c is continuous,

C2 c is decreasing,

C3 There exist constant $a \geq b > 0$, such that $c(0) - au \leq c(u) \leq c(0) - bu$, for all $u \geq 0$

C4 $c(0) > 1/2$.

The main argument, following that of [4], starts from section 1.2. First we will improve one key estimate and the required improvement is studied in section 1.1.

1.1 Some Preliminary Results

In this section, we will study the following stochastic ordinary differential equation

$$dY_t = Y_t c(Y_t)dt + Y_t dW_t \tag{1.5}$$

with initial condition $Y_0 = x > 0$. The function c satisfies condition C1 to C4 and W is the Brownian motion appearing in (1.1). The aim is to prove that for the stationary distribution $\pi(dx)$ of Y and any positive number α, there exists t_0 and a positive number $l(\alpha, p)$, such that for any $T > t_0$

$$P\left\{\left|\frac{1}{t}\int_0^t c(Y_r)dr - \int_0^\infty c(x)\pi(dx)\right| > \alpha,\ for\ any\ t > T\right\} \leq \exp(-l(\alpha,p)T^{1-p})$$

for any $p \in (0,1)$. We will mainly use the theorem on large deviation for stationary distributions of Markov processes by Donsker and Varadhan [23] and a combination of Chebechev inequality, Feyman-Kac formula and PDE theory to show this is true. We start with a series of small lemmas that will be needed before we start the main proof.

Lemma 1.1.1. *Suppose that Y satisfies the stochastic differential equation* (1.5)*. If we consider $[0,\infty)$ as the state space of Y, then it has a family of invariant probability measures on $\{[0,\infty), \mathscr{B}([0,\infty))\}$, which has the form*

$$\left\{\pi^\theta = \theta\delta_0 + (1-\theta)\pi\right\} \quad \theta \in [0,1].$$

Here δ_0 is the unit mass concentrated on point 0 and π is the unique invariant measure of Y when it is considered on the state space $(0,\infty)$.

Proof. It is obvious that δ_0 is an invariant probability measure when Y satisfying SDE in (1.5) is considered on the state space $[0,\infty)$. Therefore, if we could prove that Y has an unique invariant probability measure π when it is considered on state space $(0,\infty)$, the result of this lemma is true. To see this, we define a process $Z : [0,\infty) \to R$ by $Z_t = \ln Y_t$. Then, by Ito's formula, Z satisfies the SDE

$$dZ_t = \left(c(e^{Z_t}) - \frac{1}{2}\right) dt + dW_t. \tag{1.6}$$

Since the function c in (1.5) is continuous and bounded above, and (1.6) is in the form of a gradient system, we could conclude that there exists an unique invariant measure $\hat{\mu}_Z$ for Z, which is in the form

$$\hat{\mu}_Z(dx) = \exp\left\{2\int_0^x c(e^t)dt - x\right\} dx.$$

See e.g.[8], [33]. It is easy to see that since the function c satisfies $c(x) \leq C_0 - bx$ for some $b > 0$, there exists a positive number $M < \infty$ such that $\int_R \exp\left\{2\int_0^x c(e^t)dt - x\right\} dx = M$. Then it is obvious that there exists an unique probability measure $\mu_Z = \hat{\mu}_Z/M$ for Z.

Now if we define a map $\phi : R \to R^+ := (0,\infty)$ by

$$\phi(z) = e^z,$$

we could define the corresponding map $\phi^{-1} : \mathscr{B}(R^+) \to \mathscr{B}(R)$, which is between Borel measurable sets of R and R^+ as,

$$\phi^{-1}(\Gamma) := \{z \in R : \phi(z) \in \Gamma\} \ for \ \Gamma \in \mathscr{B}(R^+).$$

Also, we could define the μ_Z push forward measure $\phi^*\mu_Z$ on R^+ by

$$\phi^*\mu_Z(\Gamma) = \mu_Z(\phi^{-1}(\Gamma)) \ for \ \Gamma \in \mathscr{B}(R^+).$$

Now let $P_t(y, \Gamma)$ denote the semigroup of Y. For any $\Gamma \in \mathscr{B}(R^+)$, $P_t(y, \Gamma)$ is a nonnegative measurable function. Similarly, let $\hat{P}_t\left(z, \phi^{-1}(\Gamma)\right)$ denote the semigroup of Z. Then for any $\Gamma \in \mathscr{B}(R^+)$, $P_t(e^z, \Gamma) = \hat{P}_t\left(z, \phi^{-1}(\Gamma)\right)$. Then using the integration formula for push forward measure, we could see that

$$\begin{aligned}
\int_{R^+} P_t(y, \Gamma)\phi^*\mu_Z(dy) &= \int_R P_t\left(e^z, \Gamma\right)\mu_Z(dz) \\
&= \int_R \hat{P}_t(z, \phi^{-1}(\Gamma))\mu_Z(dz) \\
&= \mu_Z(\phi^{-1}(\Gamma)) = \phi^*\mu_Z(\Gamma)
\end{aligned}$$

Therefore, $\phi^*\mu_Z$ is an invariant probability measure for Y. Meanwhile, we could also see from the above equality that it is the unique one. Since if there was another invariant measure, say ν, then its pulled back measure on R^+ should be an invariant measure for the Z process different from μ_Z, which contradict with the fact that μ_Z is the unique one. Hence, we have proved that Y has a unique invariant probability measure when it is considered with state space $(0, \infty)$. Therefore the conclusion of the lemma is valid. □

Lemma 1.1.2. *Suppose that Y satisfies the stochastic differential equation* (1.5)*. Let L be the infinitesimal generator of the process having domain $\mathfrak{D}$. For any probability measure μ on $([0,\infty), \mathscr{B}([0,\infty)))$, define the function $I(\mu)$ as*

$$I(\mu) = - \inf_{u>0, u\in\mathfrak{D}} \int_{\bar{R}^+} \frac{Lu}{u}(x)\mu(dx),$$

where $\bar{R}^+ = [0,\infty)$. If $I(\mu) = 0$, then μ is an invariant probability measure for the Y process.

Proof. It is sufficient to show that if Y_0 is distributed as $\mu(dx)$, $E\left[u\left(Y_t\right)\right] = E\left[u\left(Y_0\right)\right] = \int_{\bar{R}^+} u(x)\mu(dx)$ for all $t \geq 0$, for any $u \in C_b$, the set of bounded continuous functions on $\bar{R}^+$.

From the definition of $I(\mu)$ and the hypothesis $I(\mu) = 0$, we could see that for any $u > 0$, $u \in \mathfrak{D}$, we have

$$\int_{\bar{R}^+} \frac{Lu}{u}(x)\mu(dx) \geq 0 \tag{1.7}$$

Let u be a function such that $u > 0$ and $u \in C_b \cap \mathfrak{D}$. Let C be a positive number such that $u < C$. Take ϵ to be a number such that $|\epsilon| < \frac{1}{C}$ and define a function $v = 1 + \epsilon u$. We can see that $v > 0$ and $v \in C_b \cap \mathfrak{D}$. From the observation (1.7), we have

$$\begin{aligned}
0 &\leq \int_{\bar{R}^+} \frac{Lv}{v}(x)\mu(dx) = \int_{\bar{R}^+} \frac{L(1+\epsilon u)}{1+\epsilon u}(x)\mu(dx) \\
&= \epsilon \int_{\bar{R}^+} \frac{L(u)}{1+\epsilon u}(x)\mu(dx) \\
&= \epsilon \int_{\bar{R}^+} (Lu)(1 - \epsilon u + (\epsilon u)^2 + \ldots)(x)\mu(dx) \\
&= \epsilon \int_{\bar{R}^+} (Lu)(x)\mu(dx) + O(\epsilon^2)
\end{aligned}$$

If we view the right hand side of the above inequality as a function $H(\epsilon)$, we could see the only way to make sure $H(\epsilon) \geq 0$ at an interval around 0 is $\int_{\bar{R}^+} (Lu)(x)\mu(dx) = 0$. Therefore, we have shown that for any $u > 0$ and $u \in C_b \cap \mathfrak{D}$,

$$\int_{\bar{R}^+} (Lu)(x)\mu(dx) = 0 \tag{1.8}$$

Recall that semigroup $\{T_t, t \geq 0\}$ for Y_t can be defined as follows, for any bounded measurable functions $u : R^+ \to R$

$$(T_t u)(x) = E\left[u(Y_t) | Y_0 = x\right] \tag{1.9}$$

Then, if Y_0 starts distributed as $\mu(dx)$, we could see that

$$\begin{aligned}
\frac{d}{dt}E\left[u(Y_t)\right] &= \frac{d}{dt}\left[\int_{\bar{R}^+}(T_t u)(x)\mu(dx)\right] \\
&= \int_{\bar{R}^+}\frac{d}{dt}(T_t u)(x)\mu(dx) \\
&= \int_{\bar{R}^+}L(T_t u)(x)\mu(dx)
\end{aligned}$$

The last equality is true since the semigroup $\{T_t, t \geq 0\}$ we defined is strongly continuous. For the same reason, we could see that for any $u > 0$ and $u \in C_b \cap \mathfrak{D}$, we have $T_t u > 0$ and $T_t u \in C_b \cap \mathfrak{D}$ for any $t \geq 0$. Therefore, by (1.8), we could conclude that for any positive bounded continuous function, if Y_t starts distributed as μ, we have $\frac{d}{dt}E\left[u(Y_t)\right] = 0$, for all $t \geq 0$. Through standard approximation, we can see that the same is true for any bounded continuous functions, i.e. we have proved that if Y_0 is distributed as $\mu(dx)$, $E\left[u\left(Y_t\right)\right] = E\left[u\left(Y_0\right)\right]$ for all $t \geq 0$, for any $u \in C_b$. □

Lemma 1.1.3. *Suppose that Y satisfies the stochastic differential equation* (1.5) *and $\pi(dx)$ is the unique invariant probability measure of Y when it is considered on state space $(0, \infty)$. Let $\mathfrak{M}$ be the space of probability measures on $\{0\} \cup R^+$ and endow this space with the topology of weak convergence. For any number $\alpha > 0$, define a set F_α as*

$$F_\alpha = \left\{\mu \in \mathfrak{M} : \left|\int_0^\infty f(x)\mu(dx) - \int_0^\infty f(x)\pi(dx)\right| \geq \alpha\right\}.$$

where f is a bounded continuous function. For some sequences $\{\delta_n\}$ and $\{\beta_n\}$ such that $\delta_n \downarrow 0$ and $\beta_n \downarrow 0$, define a set G as

$$G = \{\mu \in \mathfrak{M} : \mu\left([0, \delta_n]\right) \leq \beta_n \; for \; all \; n\} \tag{1.10}$$

Then $F_\alpha \cap G$ is a closed set in $\mathfrak{M}$ and for $I(\mu)$ defined as in Lemma 1.1.2, $\inf_{\mu \in F_\alpha \cap G} I(\mu) > 0$.

Proof. It is not difficult to see that $F_\alpha \cap G$ is a closed set. So we will prove that $\inf_{\mu \in F_\alpha \cap G} I(\mu) > 0$. From Lemma 1.1.2, we know that if $I(\mu) = 0$, then μ is an invariant measure for Y process. From Lemma 1.1.1, any invariant measure π^θ of Y process has the form

$$\pi^\theta = \theta\delta_0 + (1-\theta)\pi \ for \ some \ \theta \in [0,1].$$

When $\theta = 0$, it is obvious that $\pi^\theta \notin F_\alpha$ for all $\alpha > 0$. When $\theta \in (0,1]$, for any positive number δ_n, $\pi^\theta([0,\delta_n]) \geq \theta$, and hence $\pi^\theta \notin G$. Therefore the set $F_\alpha \cap G$ does not contain any invariant measure of Y process. Then we can conclude that for any $\mu \in F_\alpha \cap G$, $I(\mu) > 0$.

Let $\hat{K}$ be a compact set in $\mathfrak{M}$. For a number $l > 0$, define a set K as

$$K = \{\mu \in \mathfrak{M} : \ I(\mu) \leq l\} \cap \hat{K}$$

Since $I(\mu)$ is lower semi-continuous (see [16]), $\{\mu \in \mathfrak{M} : \ I(\mu) \leq l\}$ is a closed set. Therefore set K will be a compact set as it is the intersection of a closed set and a compact set. Let H denote the set $F_\alpha \cap G$. Then we have

$$\inf_{\mu \in H} I(\mu) = \min \left[\inf_{\mu \in (H \cap K)} I(\mu), \inf_{\mu \in (H \cap K^c)} I(\mu) \right]$$

Since lower semi-continuous function attains its infimum on compact set, we know that

$$\inf_{\mu \in (H \cap K)} I(\mu) > 0.$$

Also, since

$$\inf_{\mu \in (H \cap K^c)} I(\mu) \geq \inf_{\mu \in K^c} I(\mu) \geq l > 0,$$

we conclude that

$$\inf_{\mu \in F_\alpha \cap G} I(\mu) > 0.$$

$\square$

For the rest of this section, for any positive number N, we define c^N as the truncated function of c (appearing in (1.5)) in the following way

$$c^N(x) = \max\{c(x), -N\} \tag{1.11}$$

As $c(x)$ is a deceasing function, it is bounded above by $c(0)$. It is obvious that for any number N, c^N will be a bounded function.

Lemma 1.1.4. *Suppose that $\pi(dx)$ is the unique invariant measure of Y process satisfying stochastic differential equation in* (1.5) *when considered* *on state space* $(0,\infty)$*. Then for any* $\alpha > 0$*, there exists* N^**, such that for any* $N > N^*$*,*

$$\left| \int_0^\infty c^N(x)\pi(dx) - \int_0^\infty c(x)\pi(dx) \right| < \alpha/3$$

where c *is the function appeared in* (1.5) *and* c^N *is defined as in* (1.11)*.*

Proof. Since c is a continuous decreasing function satisfying $c(0)-ax \le c(x)$, for some $a > 0$, then for any N, there exists a positive number $M(N)$ satisfying $\lim_{N\to\infty} M(N) = \infty$, such that

$$\left| \int_0^\infty c^N(x)\pi(dx) - \int_0^\infty c(x)\pi(dx) \right| \le \int_M^\infty \left(-N - c(0) + ax\right)\pi(dx).$$

As shown in the proof in Lemma 1.1.1, $\pi(dx) = \exp\left(2\int_0^{\ln x} c(e^t)dt - \ln x\right) dx$. Since the function c satisfies $c(0)-ax \le c(x) \le c(0)-bx$, for some $a \ge b > 0$,

$$\begin{aligned}
& \int_M^\infty \left(-N - c(0) + ax\right)\pi(dx) \\
= \; & \int_M^\infty \left(-N - c(0) + ax\right)\exp\left(2\int_0^{\ln x} c(e^t)dt - \ln x\right) dx \\
\le \; & \int_M^\infty \left(-N - c(0) + ax\right)\exp\left(2\int_0^{\ln x} \left(c(0) - be^t\right) dt - \ln x\right) dx \\
= \; & \int_M^\infty \left(-N - c(0) + ax\right)\left(x^{(2c(0)-1)} e^{-2b(x-1)}\right) dx
\end{aligned}$$

Since $b > 0$ and $M \to \infty$ as $N \to \infty$,

$$\lim_{N\to\infty} \left| \int_0^\infty c^N(x)\pi(dx) - \int_0^\infty c(x)\pi(dx) \right| = 0.$$

Hence the conclusion in the lemma is valid. □

Lemma 1.1.5. *Suppose that* Y *satisfies the SDE* (1.5)*, and it starts from some fixed point* $Y_0 = x > 0$*. There exist sequences* $\{\delta_n\}$ *and* $\{\beta_n\}$ *such that* $\delta_n \to 0$*,* $\beta_n \to 0$ *as* $n \to \infty$*, and for any* $0 < p < 1$*, there exist* $h > 0$*,* $T_1 > 0$*, depending on* x*,* p *and the function* c*, such that for any* $t > T_1$*,*

$$\sum_{n=1}^\infty P_x\left(\frac{1}{t}\int_0^t \mathbf{1}_{[0,\delta_n]}(\omega_r^Y)dr > \beta_n\right) \le \exp(-ht^{1-p}),$$

where P_x is the probability measure induced by Y on the space Ω_x, the space of continuous functions $f : [0, \infty) \to (0, \infty)$, which starts from $f(0) = x > 0$.

Proof. To make the calculation later easier, we introduce the Z process defined as in the proof of Lemma 1.1.1. Then Z satisfies the SDE (1.6), and its starting value is $Z_0 = z = \ln x$. Then by Chebychev's inequality, for any $\theta > 0$,

$$\begin{aligned} & P_x\left(\frac{1}{t}\int_0^t \mathbf{1}_{[0,\delta_n]}(\omega_r^Y)dr > \beta_n\right) \qquad (1.12) \\ = \; & \hat{P}_z\left(\exp\left(\theta\int_0^t \mathbf{1}_{(-\infty,\ln\delta_n]}(\omega_r^Z)dr\right) > \exp\left(\theta\beta_n t\right)\right) \\ \leq \; & \exp\left(-\theta\beta_n t\right)\hat{E}_z\left[\exp\left(\theta\int_0^t \mathbf{1}_{(-\infty,\ln\delta_n]}(\omega_r^Z)dr\right)\right] \end{aligned}$$

where $\hat{P}_z$ is the probability measure induced by the Z process on the space Ω_z, the space of continuous functions $f : [0, \infty) \to R$, which starts from $z = \ln x$. If we define $u(t, z) = \hat{E}_z\left[\exp\left(\theta\int_0^t \mathbf{1}_{(-\infty,\ln\delta_n]}(\omega_r^Z)dr\right)\right]$, then by the Feyman-Kac formula, it satisfies the PDE

$$\begin{aligned} u_t(t, z) &= \frac{1}{2}u_{zz}(t, z) + \left(c(e^z) - \frac{1}{2}\right)u_z(t, z) \qquad (1.13) \\ &\quad +\theta\mathbf{1}_{(-\infty,\ln\delta_n]}(z)u(t, z) \\ u(0, z) &= 1 \end{aligned}$$

where this equation is considered on domain $(t, z) \in [0, \infty) \times R$. Then by comparison argument, any function $U(t, z)$ satisfying

$$\begin{aligned} U_t(t, z) &\geq \frac{1}{2}U_{zz}(t, z) + \left(c(e^z) - \frac{1}{2}\right)U_z(t, z) + \theta\mathbf{1}_{(-\infty,\ln\delta_n]}(z)U(t, z) \\ U(0, z) &\geq 1, \end{aligned}$$

is a super solution for $u(t, z)$.

Since $c(0) > \frac{1}{2}$, there exists a positive number M such that $M(c(0) - \frac{1}{2}) > 1$. Also, since c is a continuous decreasing function, there exists a number a such that $c(a) = 0$. We claim that if we take $\theta = \theta_n = \frac{1}{2M^2\sqrt{n}}$ and

$\delta_n = \exp\left(-\frac{1}{2c(0)}n^{1/p}\right)$ respectively in equation (1.13), then there exists N_0 depending on the function c and the constant p such that for all $n \geq N_0$,

$$U(t,z) = \exp(M^2\theta_n^2 t)\cosh\left(M\theta_n\left(z - \ln a\right)\right)$$

is a super solution for u satisfying (1.13) with θ_n and δ_n. To see that this claim is valid, first it is easy to see that $U(0,z) = \cosh\left(M\theta_n\left(z - \ln a\right)\right) \geq 1$, for all $z \in R$. Then through simple calculation, we could see that all we need to check is that for all $z \in R$

$$\frac{1}{2}M^2\theta_n \geq \left(c(e^z) - \frac{1}{2}\right)M\frac{\sinh\left(M\theta_n\left(z - \ln a\right)\right)}{\cosh\left(M\theta_n\left(z - \ln a\right)\right)} + \mathbf{1}_{(-\infty,\ln\delta_n]}(z). \qquad (1.14)$$

Too see that (1.14) is true, we note that the term $\left(c(e^z) - \frac{1}{2}\right)M$ and $\frac{\sinh(M\theta_n(z-\ln a))}{\cosh(M\theta_n(z-\ln a))}$ always have the opposite signs. Therefore for $z > \ln\delta_n$, the RHS of (1.14) is non-positive. So we conclude that for any $\delta_n > 0$, if $z > \ln\delta_n$, (1.14) would be true for any $\theta_n > 0$. Meanwhile, it is easy to see that when $z \leq \ln\delta_n$

$$\begin{aligned}
& \left(c(e^z) - \frac{1}{2}\right)M\frac{\sinh\left(M\theta_n\left(z - \ln a\right)\right)}{\cosh\left(M\theta_n\left(z - \ln a\right)\right)} \\
\leq & \left(c(\delta_n) - \frac{1}{2}\right)M\frac{\sinh\left(M\theta_n\left(\ln\delta_n - \ln a\right)\right)}{\cosh\left(M\theta_n\left(\ln\delta_n - \ln a\right)\right)} \\
= & \left(c(\delta_n) - \frac{1}{2}\right)M\frac{\sinh\left(\frac{1}{2M\sqrt{n}}\left(-\frac{1}{2c(0)}n^{1/p} - \ln a\right)\right)}{\cosh\left(\frac{1}{2M\sqrt{n}}\left(-\frac{1}{2c(0)}n^{1/p} - \ln a\right)\right)}.
\end{aligned}$$

We know that

$$\lim_{n\to\infty}\left(c(\delta_n) - \frac{1}{2}\right)M = \left(c(0) - \frac{1}{2}\right)M > 1.$$

Also since $0 < p < 1$, $\lim_{n\to\infty} n^{1/p-1/2} = +\infty$, then

$$\lim_{n\to\infty}\frac{\sinh\left(\frac{1}{2M\sqrt{n}}\left(-\frac{1}{2c(0)}n^{1/p} - \ln a\right)\right)}{\cosh\left(\frac{1}{2M\sqrt{n}}\left(-\frac{1}{2c(0)}n^{1/p} - \ln a\right)\right)} = -1.$$

Then we conclude that there exists N_0 depending on the function c and the constant p such that for all $n \geq N_0$, the RHS of (1.14) would be negative, and hence our claim is true.

Now, for any $n > N_0$, for $\theta = \frac{\beta_n}{2M^2} = \frac{1}{2M^2\sqrt{n}}$, (i.e. $\beta_n = \frac{1}{\sqrt{n}}$), $\delta_n = \exp\left(-\frac{1}{2c(0)}n^{1/p}\right)$, by (1.12) and the claim we just proved, we see that

$$P_x\left(\frac{1}{t}\int_0^t \mathbf{1}_{[0,\delta_n]}(\omega_r^Y)dr > \beta_n\right) \leq \exp\left(-\frac{\beta_n^2}{4M^2}t\right)\cosh\left(\frac{\beta_n}{2M}(\ln x - \ln a)\right) \tag{1.15}$$

We define the sequence of $\left\{\theta_n^{'}\right\}$ and $\left\{\beta_n^{'}\right\}$ to be $\theta_n^{'} = \theta_{n+N_0}$, $\beta_n^{'} = \beta_{n+N_0}$, and without confusion, we still use θ_n and β_n for the new shifted sequences.

Next, if we claim that

$$P_x\left(\frac{1}{t}\int_0^t \mathbf{1}_{[0,\delta_n]}(\omega_r^Y)dr > \beta_n\right) \leq \frac{\delta_n^{2c(0)}}{2c(0)\pi([0,x])\beta_n} \tag{1.16}$$

where π is the invariant probability measure for Y process when it is considered on the state space $(0,\infty)$. Assuming the claim we have that

$$\begin{aligned}
&\sum_{n=1}^{\infty} P_x\left(\frac{1}{t}\int_0^t \mathbf{1}_{[0,\delta_n]}(\omega_r^Y)dr > \beta_n\right) \qquad (1.17)\\
\leq\ &\sum_{n=1}^{\infty}\left\{\exp\left(-\frac{\beta_n^2}{4M^2}t\right)\cosh\left(\frac{\beta_n}{2M}(\ln x - \ln a)\right)\right\} \wedge \left\{\frac{\delta_n^{2c(0)}}{2c(0)\pi([0,x])\beta_n}\right\}.
\end{aligned}$$

It is not difficult to see that

$$\begin{aligned}
&\exp\left(-\frac{\beta_n^2}{4M^2}t\right)\cosh\left(\frac{\beta_n}{2M}(\ln x - \ln a)\right) \qquad (1.18)\\
\leq\ &\exp\left(-\frac{1}{4M^2}\frac{t}{n+N_0}\right)\cosh\left(\frac{1}{2M}(\ln x - \ln a)\right)\\
\leq\ &c_1(x)\exp\left(-\frac{t}{4M^2(n+N_0)}\right).
\end{aligned}$$

where for any x, $c_1(x)$ is a positive constant depending only on x and the function c. For any $0 < p < 1$, we take $\delta_n = \exp\left(-\frac{1}{2c(0)}(n+N_0)^{1/p}\right)$. Then we could see that there exists $0 < k < 1$ depending on p and a positive constant $c_2(x)$ depending on x and the function c such that for all n,

$$\frac{\delta_n^{2c(0)}}{2c(0)\pi([0,x])\beta_n} = \frac{\sqrt{n+N_0}\exp\left(-(n+N_0)^{1/p}\right)}{2c(0)\pi([0,x])} \leq c_2(x)\exp\left(-k(n+N_0)^{1/p}\right). \tag{1.19}$$

Using (1.18) and (1.19), we could continue (1.17) to see that for t large enough,

$$\begin{aligned}
&\sum_{n=1}^{\infty} P_x\left(\frac{1}{t}\int_0^t \mathbf{1}_{[0,\delta_n]}(\omega_r^Y)dr > \beta_n\right) \qquad (1.20)\\
\leq\ & \sum_{n=1}^{[t^p]} \exp\left(-\frac{\beta_n^2}{4M^2}t\right)\cosh\left(\frac{\beta_n}{2M}(\ln x - \ln a)\right) + \sum_{n=[t^p]+1}^{\infty} \frac{\delta_n^{2c(0)}}{2c(0)\pi([0,x])\beta_n}\\
\leq\ & \sum_{n=1}^{[t^p]} c_1(x)\exp\left(-\frac{t}{4M^2(n+N_0)}\right) + \sum_{n=[t^p]+1}^{\infty} c_2(x)\exp\left(-k(n+N_0)^{1/p}\right)\\
\leq\ & t^p c_1(x)\exp\left(-\frac{1}{4M^2}t^{1-p}\right) + c_2(x)\exp(-kt)
\end{aligned}$$

Therefore, we could see that for any fixed $x > 0$, for any $0 < p < 1$, there exists T_1 and $h > 0$, depending on x, p, and the function c such that for $t > T_1$,

$$\sum_{n=1}^{\infty} P_x\left(\frac{1}{t}\int_0^t \mathbf{1}_{[0,\delta_n]}(\omega_r^Y)dr > \beta_n\right) \leq \exp\left(-ht^{1-p}\right).$$

i.e. the conclusion of this lemma is valid. So the rest of this proof will be used to show that (1.16) is true. To see this we introduce another process $\bar{Y}$ which satisfies the same SDE as Y and is driven by the same Brownian motion on the same probability space. The difference between Y and $\bar{Y}$ is that $\bar{Y}_0$ is random and distributed according to π, the unique invariant probability measure of Y when it is considered on state space $(0,\infty)$, where as $Y_0 = x > 0$ Then use the Markov property and the comparison theorem for SDE, we could see that for any $\delta_n > 0$,

$$\begin{aligned}
P\left(\bar{Y}_t \in [0,\delta_n]\right) &\geq P\left(\bar{Y}_0 \leq x,\ \bar{Y}_t \in [0,\delta_n]\right) \geq P\left(\bar{Y}_0 \leq x,\ Y_t \in [0,\delta_n]\right)\\
&= \pi([0,x])\,P(Y_t \in [0,\delta_n]).
\end{aligned}$$

Now we use Chebychev's inequality again to see that

$$\begin{aligned}
& P_x\left(\frac{1}{t}\int_0^t \mathbf{1}_{[0,\delta_n]}(\omega_r^Y)dr > \beta_n\right) \leq \frac{1}{\beta_n}E_x\left[\frac{1}{t}\int_0^t \mathbf{1}_{[0,\delta_n]}(\omega_r^Y)dr\right] \\
\leq\ & \frac{1}{\beta_n\pi\left([0,x]\right)}E_\pi\left[\frac{1}{t}\int_0^t \mathbf{1}_{[0,\delta_n]}(\omega_r^{\bar{Y}})dr\right] \\
\leq\ & \frac{1}{\beta_n\pi\left([0,x]\right)}\int_0^{\delta_n}\exp\left(2\int_0^{\ln x}c(e^t)dt-\ln x\right)dx \\
\leq\ & \frac{1}{\beta_n\pi\left([0,x]\right)}\int_0^{\delta_n}x^{2c(0)-1}dx \\
=\ & \frac{\delta_n^{2c(0)}}{2c(0)\pi([0,x])\beta_n}.
\end{aligned}$$

□

Lemma 1.1.6. *Suppose that Y satisfies the stochastic differential equation* (1.5) *and it starts from $Y_0 = x > 0$. Let $\pi(dx)$ be the unique invariant probability measure when Y is considered on state space $(0,\infty)$. Then for any $p \in (0,1)$, any $\alpha > 0$, any $N > 0$, there exists a number $I(\alpha,p) > 0$ which also depends on x, N, the function c, and T^* depending on x, p, N,α and the function c, such that for any $t > T^*$,*

$$P_x\left\{\left|\frac{1}{t}\int_0^t c^N(Y_r)dr - \int_0^\infty c^N(x)\pi(dx)\right| > \frac{\alpha}{3}\right\} \leq \exp\left(-I(\alpha,p)t^{1-p}\right)$$

Proof. As in Donsker and Varadhan's work [23], for each $t > 0$, $\omega \in \Omega_x$, and any set $A \in \mathscr{B}\left([0,+\infty)\right)$, define

$$L_t(\omega,A) = \frac{1}{t}\int_0^t \mathbf{1}_A(\omega_r)dr$$

Recall that in Lemma 1.1.3, we defined $\mathfrak{M}$ to be the space of probability measures on $[0,\infty)$ with the topology of weak convergence. We could see that for each $t > 0$ and $\omega \in \Omega_x$, $L_t(\omega,\cdot) \in \mathfrak{M}$. Then we could define a measure $Q_{x,t}$ on $\mathfrak{M}$ by $Q_{x,t} = P_x L_t^{-1}$, i.e. for any Borel set $B \subset \mathfrak{M}$, $Q_{x,t}(B) = P_x\left\{\omega \in \Omega_x :\ L_t(\omega,\cdot) \in B\right\}$. Then we could see that for function c^N defined in (1.11), define a subset of $\mathfrak{M}$ as

$$C_{N,\alpha} = \left\{\mu \in \mathfrak{M} :\ \left|\int_0^\infty c^N(x)\mu(dx) - \int_0^\infty c^N(x)\pi(dx)\right| \geq \frac{\alpha}{3}\right\}.$$

Then

$$
\begin{aligned}
& P_x\left\{\left|\frac{1}{t}\int_0^t c^N(Y_r)dr - \int_0^\infty c^N(x)\pi(dx)\right| \geq \frac{\alpha}{3}\right\} \qquad (1.21)\\
= \; & P_x\left\{\left|\int_0^\infty c^N(x)L_t(\omega,dx) - \int_0^\infty c^N(x)\pi(dx)\right| \geq \frac{\alpha}{3}\right\}\\
= \; & Q_{x,t}(C_{N,\alpha})\\
\leq \; & Q_{x,t}\left(C_{N,\alpha}\cap G\right) + Q_{x,t}\left(G^c\right)
\end{aligned}
$$

where G is defined as in (1.10) with $\{\delta_n\}$, $\{\beta_n\}$ chosen as in Lemma 1.1.5. It is not difficult to see that

$$
Q_{x,t}\left(G^c\right) \leq \sum_{n=1}^{\infty} P_x\left(\frac{1}{t}\int_0^t \mathbf{1}_{[0,\delta_n]}(\omega_r^Y)dr > \beta_n\right).
$$

Therefore by Lemma 1.1.5, for any $p\in(0,1)$, there exists $h>0$, T_1, such that for any $t>T_1$

$$
Q_{x,t}\left(G^c\right) \leq \exp\left(-ht^{1-p}\right). \qquad (1.22)
$$

All we need to do is use the upper bound Donsker and Varadhan proved in [23] to estimate $Q_{x,t}(C_{N,\alpha}\cap G)$. First we need to check the following hypothesis in their theorem: There must exist a function $V(x)$ on $[0,\infty)$ such that $\{x\in[0,\infty):\ V(x)\geq h\}$ is a compact set for each $h>-\infty$ and a sequence $\{u_n(x)\}\in\mathfrak{D}$ satisfying the following conditions, where $\mathfrak{D}$ is the domain of Y process' infinitesimal generator,

1. $u_n(x)\geq 1$ for all n and $x\in[0,\infty)$;
2. for each compact set $W\subset[0,\infty)$,

$$
\sup_{x\in W}\sup_n u_n(x) < \infty;
$$

3. for each $x\in[0,\infty)$,

$$
\lim_{n\to\infty}\left(\frac{Lu_n}{u_n}\right)(x) = V(x);
$$

4. for some $A<\infty$,

$$
\sup_{n,x}\left(\frac{Lu_n}{u_n}\right)(x) \leq A.
$$

To verify this hypothesis, we define the sequence $\{u_n(x)\}$ as

$$u_n(x) = \begin{cases} x^p + 1 & 0 \le x \le n; \\ 2n & x \ge 2n; \\ \textit{smooth and increasing} & n < x < 2n \\ \textit{having bounded } 1st \textit{ and } 2nd \textit{ order derivatives} & \end{cases} \tag{1.23}$$

the first and second conditions are satisfied. Also, for this sequence

$$\begin{aligned} \lim_{n\to\infty} \frac{Lu_n}{u_n}(x) &= \lim_{n\to\infty} \frac{\frac{1}{2}x^2 u_n''(x) + xc(c)u_n'(x)}{u_n(x)} \\ &= \frac{\frac{1}{2}p(p-1)x^p + c(x)px^p}{x^p+1} \\ &:= V(x) \end{aligned}$$

Since $\lim_{x\to\infty} c(x) = -\infty$, the set $\{x: V(x) \ge h\}$ for $h > -\infty$ is compact. Finally, the last condition is satisfied by taking $A = \frac{1}{2}p(p-1) + c(0)p$. Now we can use the upper bound in the large deviation theorem which says that for any compact set $W \subset [0,\infty)$, any closed set C

$$\limsup_{t\to\infty} \frac{1}{t} \ln \sup_{x\in W} Q_{x,t}(C) \le - \inf_{\mu \in C} I(\mu)$$

where $I(\mu)$ is defined as in Lemma 1.1.2. Therefore, for any $\alpha > 0$, any N, by Lemma 1.1.3, the set $C_{N,\alpha} \cap G$ is a closed set, $\hat{I}(\alpha) := \inf_{\mu \in C_{N,\alpha}\cap G} I(\mu) > 0$ and there exists T_2 such that for any $t > T_2$,

$$Q_{x,t}\left(C_{N,\alpha} \cap G\right) \le \exp\left(-\hat{I}(\alpha)t\right). \tag{1.24}$$

Then, consider (1.21), (1.22) and (1.24) together, we could see that for any $p \in (0,1)$, there exists $I(\alpha,p) > 0$, $T^* = \max(T_1,\ T_2)$, such that for any $t > T^*$,

$$P_x\left\{\left|\frac{1}{t}\int_0^t c^N(Y_r)dr - \int_0^\infty c^N(x)\pi(dx)\right| > \frac{\alpha}{3}\right\} \le \exp\left(-I(\alpha,p)t^{1-p}\right).$$

□

Lemma 1.1.7. *Suppose that Y satisfies the stochastic differential equation*

(1.5) *and function* c^N *is defined as in* (1.11). *Then for any fixed initial condition* $x > 0$, *any number* $\alpha > 0$, *any* $l \in (0,1)$, *there exists* N^{**} *depending on* α *and the function* c, T^{**} *depending on* x, l *and function* c, *such that for any* $N > N^{**}$, $t > T^{**}$,

$$P_x \left\{ \left| \int_0^t c(\omega_r) dr - \int_0^t c^N(\omega_r) dr \right| > \frac{\alpha t}{3} \right\} \leq \exp(-lt)$$

Proof. From the definition of c^N, we could see that

$$\left| \int_0^t c(\omega_r) dr - \int_0^t c^N(\omega_r) dr \right| = \int_0^t c^N(\omega_r) - c(\omega_r) dr := \int_0^t f^N(\omega_r) dr,$$

where $f^N(x) = c^N(x) - c(x)$. Since c is a decreasing function, for any N, there exists $M > 0$, such that $f^N(x) = \mathbf{1}_{(x \geq M)}(-N - c(x))$, and $M \to \infty$ as $N \to \infty$. Then by Chebychev's inequality, for any $\theta > 0$, we have

$$\begin{aligned} & P_x \left\{ \left| \int_0^t c(\omega_r) dr - \int_0^t c^N(\omega_r) dr \right| > \frac{\alpha t}{3} \right\} \\ = \; & P_x \left\{ \exp\left(\theta \int_0^t f^N(\omega_r) dr \right) \geq \exp\left(\frac{\theta \alpha t}{3} \right) \right\} \\ \leq \; & \exp\left(-\frac{\theta \alpha t}{3} \right) E_x \left[\exp\left(\int_0^t \theta f^N(\omega_r) dr \right) \right]. \end{aligned} \tag{1.25}$$

If we define $\psi^N(t,x) = E_x \left[\exp\left(\int_0^t \theta f^N(\omega_r) dr \right) \right]$, then by Feyman-Kac formula, $\psi^N(t,x)$ satisfies the PDE

$$\begin{aligned} \partial_t \psi^N(t,x) &= \frac{1}{2} x^2 \partial_{xx} \psi^N(t,x) + x c(x) \partial_x \psi^N(t,x) + \psi^N(t,x) \theta f^N(x), \\ \psi^N(0,x) &= 1. \end{aligned}$$

Then by a comparison theorem for parabolic PDE, any function $U^N(t,x)$

satisfying

$$\begin{aligned}\partial_t U^N(t,x) &\geq \frac{1}{2}x^2\partial_{xx}U^N(t,x) + xc(x)\partial_x U^N(t,x) \\ &\quad + U^N(t,x)\theta f^N(x), \\ U^N(0,x) &\geq 1,\end{aligned} \tag{1.26}$$

is a super solution for $\psi^N(t,x)$, i.e. $\psi^N(t,x) \leq U^N(t,x)$ for all $(t,x) \in [0,\infty)\times(0,\infty)$. If we take $U^N(t,x) = \exp\left(\beta^N t + k^N x\right)$, then (1.26) becomes

$$\begin{aligned}\frac{1}{2}(k^N)^2x^2 + xc(x)k^N + \theta f^N(x) - \beta^N &\leq 0, \\ k^N &\geq 0.\end{aligned}$$

Since function c satisfies the condition that $c(0) - ax \leq c(x) \leq c(0) - bx$, for some $a \geq b > 0$, then

$$\begin{aligned}& \frac{1}{2}(k^N)^2x^2 + xc(x)k^N + \theta f^N(x) - \beta^N \\ \leq\ & \frac{1}{2}(k^N)^2x^2 + k^N x\left(c(0) - bx\right) + \theta \mathbf{1}_{(x\geq M)}\left(-N - c(0) + ax\right) - \beta^N \\ :=\ & H(x).\end{aligned}$$

Taking $k^N = b$, and $\beta^N = c^2(0)/2$, it is not difficult to see that for any $\theta > 0$, there exists N_1, such that for any $N > N_1$, $H(x) \leq 0$ for all $x > 0$, and hence that $\exp\left(\frac{c^2(0)}{2}t + bx\right) \geq \psi^N(t,x)$.

Now back to (1.25), by taking $\theta = \frac{3c^2(0)}{2\alpha} + \frac{3}{\alpha}$, we could see that there exists N^{**}, such that for any $N > N^{**}$

$$\begin{aligned}& P_x\left\{\left|\int_0^t c(\omega_r)dr - \int_0^t c^N(\omega_r)dr\right| > \frac{\alpha t}{3}\right\} \\ \leq\ & \exp\left(-\frac{\theta\alpha t}{3}\right)\psi^N(t,x) \\ \leq\ & \exp\left(-\frac{\theta\alpha t}{3} + \frac{c^2(0)}{2}t + bx\right) \\ =\ & \exp\left(-t + bx\right).\end{aligned} \tag{1.27}$$

For fixed initial condition x, for any $l \in (0,1)$, there exists T^{**}, such that for $t > T^{**}$, $1 - (bx/t) > l$. Therefore we have shown that for any $\alpha > 0$, for any $l \in (0, 1)$, there exists N^{**}, T^{**} such that for any $N > N^{**}$, $t > T^{**}$,

$$P_x \left\{ \left| \int_0^t c(\omega_r) dr - \int_0^t c^N(\omega_r) dr \right| > \frac{\alpha t}{3} \right\} \le \exp(-lt)$$

$\square$

Lemma 1.1.8. *Suppose that Y satisfies the stochastic differential equation (1.5) and it starts from $Y_0 = x > 0$. Let $\pi(dx)$ be the invariant probability measure of Y when it is considered on state space $[0, \infty)$. Then for any $\alpha > 0$, for any $p \in (0, 1)$, there is a number $h(\alpha, p) > 0$ and $t_0 > 0$ such that for any $T > t_0$,*

$$\begin{aligned} & P\left\{ \left| \frac{1}{t} \int_0^t c(Y_r) dr - \int_0^\infty c(x) \pi(dx) \right| > \alpha \ for\ all\ t > T \right\} \\ \le\ & \exp(-h(\alpha, p) T^{1-p}) \end{aligned}$$

Proof. It is easy to see that

$$\begin{aligned} & \left| \frac{1}{t} \int_0^t c(Y_r) dr - \int_0^\infty c(x) \pi(dx) \right| \\ \le\ & \left| \frac{1}{t} \int_0^t c(Y_r) dr - \frac{1}{t} \int_0^t c^N(Y_r) dr \right| + \left| \frac{1}{t} \int_0^t c^N(Y_r) dr - \int_0^\infty c^N(x) \pi(dx) \right| \\ & + \left| \int_0^\infty c^N(x) \pi(dx) - \int_0^\infty c(x) \pi(dx) \right|. \end{aligned}$$

Let

$$A_{\alpha,N,t} = \left\{ \left| \frac{1}{t} \int_0^t c(Y_r) dr - \frac{1}{t} \int_0^t c^N(Y_r) dr \right| > \frac{\alpha}{3} \right\}$$

$$B_{\alpha,N,t} = \left\{ \left| \frac{1}{t} \int_0^t c^N(Y_r) dr - \int_0^\infty c^N(x) \pi(dx) \right| > \frac{\alpha}{3} \right\}$$

$$C_{\alpha,t} = \left\{ \left| \frac{1}{t} \int_0^t c(Y_r) dr - \int_0^\infty c(x) \pi(dx) \right| > \alpha \right\}$$

By Lemma 1.1.7, Lemma 1.1.6 and Lemma 1.1.4, we could see that for any $\alpha > 0$, any $l \in (0, 1)$, any $p \in (0, 1)$, taking $N > \max(N^*, N^{**})$, there exists $T = \max(T^*,\ T^{**})$, $I(\alpha, p) > 0$, and hence $\hat{h}(\alpha, p) > 0$ such that for any

$t > T$,

$$\begin{aligned} P(C_{\alpha,t}) &\leq P(A_{\alpha,N,t}) + P(B_{\alpha,N,t}) \\ &\leq \exp(-lt) + \exp(-I(\alpha,p)t^{1-p}) \\ &\leq \exp(-\hat{h}(\alpha,p)t^{1-p}). \end{aligned}$$

Then, it is obvious that there is $h(\alpha,p) > 0$, such that

$$P(\cup_{t>T,t\in\mathbb{Z}} C_{\alpha,t}) \leq \Sigma_{t>T,t\in\mathbb{Z}} P(C_{\alpha,t}) \leq \exp(-h(\alpha,p)T^{1-p})$$

If we could show that there exists t_0, such that for $T > t_0$

$$P(\cup_{t>T} C_{\alpha,t}) \leq P(\cup_{t>T,t\in\mathbb{Z}} C_{\alpha/2,t}) \tag{1.28}$$

then the conclusion of the lemma is obviously right. So, the rest of this proof will be used to show (1.28) is true.

Suppose that for $t = M$ and $t = M + 1$, where M is an integer, we have

$$\left| \frac{1}{t} \int_0^t c(Y_r)dr - \int_0^\infty c(x)\pi(dx) \right| \leq \alpha/2$$

Then for any $t \in [M, M+1]$,

$$\begin{aligned} & \frac{1}{t} \int_0^t c(Y_r)dr - \int_0^\infty c(x)\pi(dx) \\ = \;& \frac{M}{t} \left(\frac{1}{M} \int_0^M c(Y_r)dr \right) + \frac{1}{t} \int_M^t c(Y_r)dr - \int_0^\infty c(x)\pi(dx) \\ \leq \;& \frac{M}{t} \left(\frac{1}{M} \int_0^M c(Y_r)dr - \int_0^\infty c(x)\pi(dx) \right) + \frac{c(0)}{t} + \frac{M-t}{t} \int_0^\infty c(x)\pi(dx) \\ \leq \;& \frac{M}{t}\alpha/2 + \frac{c(0)}{t} + \frac{M-t}{t} \int_0^\infty c(x)\pi(dx) \\ \leq \;& \frac{M}{t}\alpha/2 + \frac{c(0)}{t} \end{aligned}$$

and similarly,

$$
\begin{aligned}
& \frac{1}{t}\int_0^t c(Y_r)dr - \int_0^\infty c(x)\pi(dx) \\
= \ & \frac{M+1}{t}\left(\frac{1}{M+1}\int_0^{M+1} c(Y_r)dr\right) - \frac{1}{t}\int_t^{M+1} c(Y_r)dr - \int_0^\infty c(x)\pi(dx) \\
\geq \ & \frac{M+1}{t}\left(\frac{1}{M+1}\int_0^{M+1} c(Y_r)dr - \int_0^\infty c(x)\pi(dx)\right) - \frac{c(0)}{t} - \frac{t-M-1}{t}\int_0^\infty c(x)\pi(dx) \\
\geq \ & -(\alpha/2)\frac{M+1}{t} - \frac{c(0)}{t} - \frac{t-M-1}{t}\int_0^\infty c(x)\pi(dx) \\
\geq \ & -(\alpha/2)\frac{M+1}{t} - \frac{c(0)}{t}
\end{aligned}
$$

Therefore, for t large enough, we could see that for $t \in [M, M+1]$,

$$
\left|\frac{1}{t}\int_0^t c(Y_r)dr - \int_0^\infty c(x)\pi(dx)\right| \leq \alpha
$$

Thus, there exists t_0, such that for $T > t_0$

$$
P\left(\cap_{t>T, t\in\mathbb{Z}} C^c_{\alpha/2,t}\right) \leq P\left(\cap_{t>T} C^c_{\alpha,t}\right)
$$

and hence we have proved

$$
P(\cup_{t>T} C_{\alpha,t}) \leq P(\cup_{t>T,t\in\mathbb{Z}} C_{\alpha/2,t})
$$

□

Remark 1.1.9. After proving Lemma 1.1.1, Lemma 1.1.2 and Lemma 1.1.3 for Y process satisfying (1.5), we could see that the same conclusions are true if we correspondingly replace function c and c^N appeared in Lemma 1.1.4-Lemma 1.1.8 by $\hat{c}(x) = c\left((1-\lambda/2)\,x\right)$ and $\hat{c}^N(x) = \max\{\hat{c}(x), -N\}$ for any $0 < \lambda < 1$.

Here is the actual form of the above estimate that we will use later.

Lemma 1.1.10. *Suppose that Y satisfies the stochastic differential equation (1.5), and it starts from $Y_0 = x > 0$. Given fixed number $\gamma > 0$, let $0 < h < \gamma$ and $0 < \lambda < 1$ be two arbitrary numbers. For any $p \in (0,1)$, $\hat{h} \in (0, h/\gamma)$,* *if $s > \hat{h}t$, then there exist $\beta > 0$, $t_0 > 0$ and $l > 0$, such that for all $T \geq t_0$,*

$$P\left\{\int_{t-s}^{t} c\left((1-\lambda/2)Y_r\right) - c\left(Y_r\right) dr > \beta\hat{h}t \;\; \text{for all } t > T\right\} > 1 - \exp\left(-lT^{1-p}\right).$$

Proof. Since $s > \hat{h}t$, the function c is decreasing and $Y_r > 0$,

$$\int_{t-s}^{t}\left[c\left((1-\lambda/2)Y_r\right) - c\left(Y_r\right)\right] dr > \int_{t(1-\hat{h})}^{t}\left[c\left((1-\lambda/2)Y_r\right) - c\left(Y_r\right)\right] dr$$

Let π be the unique invariant probability measure of Y process when it is considered on state space $(0, \infty)$. Also because function c is decreasing, we have

$$d := \frac{1}{4}\hat{h}\int_{0}^{+\infty}\left[c\left((1-\lambda/2)x\right) - c\left(x\right)\right]\pi(dx) > 0.$$

Take $\alpha \in (0, d)$, then there exists a positive number β such that

$$0 < \beta < \int_{0}^{+\infty}\left[c\left((1-\lambda/2)x\right) - c\left(x\right)\right]\pi(dx) - \frac{4\alpha}{\hat{h}}.$$

By Lemma 1.1.8, for this α, any $p \in (0,1)$, there exists t_0^1, $h_1(\alpha, p) > 0$ and set Ω_T^1 such that for all $T > t_0^1$, $P(\Omega_T^1) \leq \exp\left\{-h_1(\alpha,p)T^{1-p}\right\}$ and for any $\omega \in \left(\Omega_T^1\right)^c$,

$$\begin{aligned} \left|\int_0^{t(1-\hat{h})} c(Y_r)dr - t(1-\hat{h})\int_0^{\infty} c(x)\pi(dx)\right| &< \alpha t(1-\hat{h}) \\ \left|\int_0^{t} c(Y_r)dr - t\int_0^{\infty} c(x)\pi(dx)\right| &< \alpha t \end{aligned}$$

for all $t(1-\hat{h}) > T$. Therefore, for such ω, and all $t(1-\hat{h}) > T$, we have

$$\begin{aligned} & \left|\int_{t(1-\hat{h})}^{t} c(Y_r)dr - t\hat{h}\int_0^{\infty} c(x)\pi(dx)\right| \\ = & \left|\int_0^{t} c(Y_r)dr - t\int_0^{\infty} c(x)\pi(dx) - \left[\int_0^{t(1-\hat{h})} c(Y_r)dr - t\left(1-\hat{h}\right)\int_0^{\infty} c(x)\pi(dx)\right]\right| \\ < & \; 2\alpha t. \end{aligned}$$

Similarly, from Remark 1.1.9, for such α and any $p \in (0,1)$ there exists t_0^2, $h_2(\alpha,p) > 0$ and set Ω_T^2, such that for all $T > t_0^2$, $P(\Omega_T^2) \leq \exp\left\{-h_2(\alpha,p)T^{1-p}\right\}$, and for any $\omega \in \left(\Omega_T^2\right)^c$,

$$\left|\int_{t(1-\hat{h})}^{t} c\left((1-\lambda/2)Y_r\right)dr - t\hat{h}\int_0^\infty c\left((1-\lambda/2)x\right)\pi(dx)\right| < 2\alpha t$$

for all $t(1-\hat{h}) > T$. Now, let $t_0 = \max\left\{t_0^1, t_0^2\right\}$, $\Omega_T = \left(\Omega_T^1\right)^c \cap \left(\Omega_T^2\right)^c$. First, it is evident that there exists $l(\alpha,p) > 0$, such that for all $T > t_0$, $P(\Omega_T) > 1-\exp(-l(\alpha,p)t^{1-p})$. Also, for $\omega \in \Omega_T$, for any $t(1-\hat{h}) > T > t_0$,

$$\begin{aligned}
&\int_{t(1-\hat{h})}^{t} \left[c\left((1-\lambda/2)Y_r\right) - c\left(Y_r\right)\right]dr \\
\geq\ & t\hat{h}\int_0^\infty c\left((1-\lambda/2)x\right)\pi(dx) - 2\alpha t - \left[t\hat{h}\int_0^\infty c(x)\pi(dx) + 2\alpha t\right] \\
=\ & t\hat{h}\left\{\int_0^\infty \left[c\left((1-\lambda/2)x\right) - c(x)\right]\pi(dx) - \frac{4\alpha}{\hat{h}}\right\} \\
>\ & t\hat{h}\beta.
\end{aligned}$$

Therefore, we have shown that given any fixed $\gamma > 0$, if $h \in (0,\gamma)$, $\lambda \in (0,1)$ are two arbitrary numbers, then for $p \in (0,1)$, $\hat{h} \in (0, h/\gamma)$, $s > \hat{h}t$, there exist $\beta > 0$, $t_0 > 0$ and $l > 0$, such that for all $T \geq t_0$,

$$P\left\{\int_{t-s}^{t} c\left((1-\lambda/2)Y_r\right) - c\left(Y_r\right)dr > \beta\hat{h}t \ \ for\ all\ t > T\right\} > 1-\exp\left(-lT^{1-p}\right).$$

$\square$

Finally, at the end of this section, we state here the lemma proved in [4] on the bound for $\lim_{t\to\infty}\frac{1}{t}\int_0^t Y_s ds$, where Y satisfies the stochastic differential equation (1.5). In the next section, we will use the method in [4] to prove an comparison theorem between the solution to the stochastic generalized KPP equation (1.1) and the Y process and use this lemma to deduce the lower bound for $\frac{1}{t}\int_0^t u(s,x)ds$ as $t\to\infty$ on the region $x \leq (\gamma-h)t$.

Lemma 1.1.11. *Suppose that Y process satisfies the stochastic differential equation* (1.5)*. Then for almost all ω,*

$$\frac{1}{a}\left(c(0)-\frac{1}{2}\right) \leq \liminf_{t\to\infty}\frac{1}{t}\int_0^t Y_s ds \leq \limsup_{t\to\infty}\frac{1}{t}\int_0^t Y_s ds \leq \frac{1}{b}\left(c(0)-\frac{1}{2}\right).$$

1.2 Improvement

In this section, we will follow the method in [4] and study

$$\partial_t u(t,x) = \frac{D}{2}\triangle u(t,x) + u(t,x)c\left(u(t,x)\right) + u(t,x)\dot{W}_t \tag{1.29}$$

with initial condition $u(0,x) = u_0(x) = \mathbf{1}_{(-\infty,0]}(x)$, for $x \in R$ and $t \in [0,\infty)$, where D is a positive constant, W is a Brownian motion on the probability space $(\Omega, \mathscr{F}, P)$. We assume that function c satisfies conditions C1 to C4.

We will study the random travelling waves for equation (1.29). As proved in [14],[19] and [3] and pointed out in [4], the equation (1.29) has a random travelling wave solution. The wavefront was known as $x = \gamma t$, where $\gamma = \sqrt{D\left(2C(0)-1\right)}$ is the wave speed. Also, it was proved in [14],[19] and [3] that there are constants $d_1 > 0$ $d_2 > 0$ and $d_3 > 0$, such that for any $h > 0$,

$$\frac{1}{t}\ln u(t,x) < -d \;\; for \;\; x > (\gamma + h)t \;\; a.s \tag{1.30}$$

and

$$-d_3 \leq \frac{\ln u(t,x)}{\sqrt{2t\ln\ln t}} \leq d_2 \;\; for \;\; x < (\gamma - h)t \;\; a.s. \tag{1.31}$$

for t large enough. As there is random term appearing in the equation, we will study

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t u(r,(\gamma+h)r)dr$$

and

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t u(r,(\gamma-h)r)dr$$

for $h > 0$ is an arbitrary number. These two quantity describe how the wave behaves ahead and behind the wavefront. From (1.30), we can see that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \sup_{x\geq(\gamma+h)t} u(s,x)ds = 0 \;\; a.s.$$

But behind the wavefront, the solution u is oscillatory. We will do the same as in [4] to study $\lim_{t\to\infty}\frac{1}{t}\int_0^t \inf_{x\leq(\gamma-h)t} u(s,x)ds$ and $\lim_{t\to\infty}\frac{1}{t}\int_0^t \sup_x u(s,x)ds$.

We will use the following stochastic Feynman-Kac formula many times in the proof. It was proved in [7], [36] and [5].

Lemma 1.2.1. *Suppose that $u(t,x)$ satisfies* (1.29). *Then there exists standard Brownian motion B on a probability space $\left(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P}\right)$, which is independent of the Brownian motion W appeared in* (1.29). *And $u(t,x)$ satisfies*

$$u(t,x) = \hat{E}\left\{u(0, x+\sqrt{D}B_t)\exp\left[\int_0^t c\left(u(t-r, x+\sqrt{D}B_r)\right)dr - \frac{1}{2}t + W_t\right]\right\} \tag{1.32}$$

where $\hat{E}$ is the expectation with respect to the probability measure $\hat{P}$.

First we can quote without proof the result about the upper bound $\lim_{t\to\infty}\frac{1}{t}\int_0^t \sup_x u(s,x)ds$ from [4].

Lemma 1.2.2. *Suppose that function c satisfies condition C1-C4 and $u(t,x)$ satisfies* (1.29). *Then for almost all $\omega \in \Omega$,*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \sup_x u(s,x)ds \le \frac{1}{b}\left(c(0) - \frac{1}{2}\right)$$

The proof of lower bound $\lim_{t\to\infty}\frac{1}{t}\int_0^t \inf_{x\le(\gamma-h)s} u(s,x)ds$ is more complicated. In order to do that, we need the following two lemmas. The first one is a special case of Lemma 3.2 in [4]. The second one is proved using methods essentially the same as methods used in [4]. This method was first used by Freidlin in proving the existance of wavefronts in reaction-diffusion equations [27]. The structure of our proof is the same as that of the proof of Lemma 3.3 in [4]. We will use the result we proved in section 1.1 to get a lower bound for

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \inf_{x\le(\gamma-h)s} u(s,x)ds$$

which is an improvement from the result in [4], in which they get the lower bound for

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \inf_{x\le(\frac{b}{a}\gamma-h)s} u(s,x)ds.$$

In the case when constants a and b differ from each other, their result only gives the lower bound for $\lim_{t\to\infty}\frac{1}{t}\int_0^t u(s,x)ds$ for x far behind the wavefront. Although in both our result and in [4], these limits are true almost surely, in [4], the rate on which the probability converges to one is exponential and in our result, it is slower than exponential. (This point is made clear later in Theorem 1.2.5)

Lemma 1.2.3. *Suppose that $u(t,x)$ satisfies* (1.29) *with initial condition $u(0,x) = u_0(x) = \mathbf{1}_{(-\infty,0]}(x)$. Assume that function c satisfies condition C1-C4. Let $\gamma = \sqrt{D\,(2c(0)-1)}$ be the wave speed. Then for any $\epsilon > 0$, there exists $t > 0$, and $\Omega_T \subset \Omega$ for all $T > t$ with $P(\Omega_T) > 1 - \exp(-\delta T)$ for a constant $\delta > 0$ and if $\omega \in \Omega_T$, then*

$$e^{-\epsilon s} \leq u(s, \gamma s) \leq e^{\epsilon s} \ \textit{for all } s \geq T$$

i.e. For almost all $\omega \in \Omega$,

$$\lim_{t\to\infty} \frac{1}{t} \ln u\,(t, \gamma t) = 0$$

Proof. We can see the result is true from Lemma 3.2 in [4] by taking $k(t) = 1$ for the function k which appeared in that Lemma. □

Lemma 1.2.4. *Suppose that function c satisfies condition C1-C4, $u(t,x)$ satisfies* (1.29) *with initial condition $u(0,x) = u_0(x) = \mathbf{1}_{(-\infty,0]}(x)$ and Y satisfies the stochastic differential equation in* (1.5) *with initial condition $Y_0 = 1$. Let $h > 0$ be an arbitrary positive number, for any fixed $p \in (0,1)$, $\lambda \in (0,1)$, there exists $t_0 > 0$ and $l > 0$ depending on p, λ and h, such that for any $T \geq t_0$,*

$$P\left\{\inf_{x<(\gamma-h)t} u(t,x) \geq (1-\lambda)Y_t, \ \textit{for any } t \geq T\right\} > 1 - \exp(-lT^{1-p}),$$

where $\gamma = \sqrt{D\,(2c(0)-1)}$.

Proof. Without loss we may take $h \in \left(0, \frac{1}{2}\gamma\right)$. Now, for fixed h, $\lambda \in (0,1)$, $p \in (0,1)$, let $\hat{h}$ be a number such that $\hat{h} \in \left(0, \frac{h}{\gamma}\right)$. From Lemma 1.1.10, we could see that for any $\lambda > 0$, there exists $\beta > 0$, $t_1 > 0$ and set $\Omega_T^1 \subset \Omega$ for all $T \geq t_1$ with $P(\Omega_T^1) > 1 - \exp(-\delta_1 T^{1-p})$ for some $\delta_1 > 0$, and if $\omega \in \Omega_T^1$, $s > \hat{h}t$, then

$$\int_{t-s}^{t} c\left((1-\lambda/2)Y_r\right) - c\left(Y_r\right) dr > \beta\hat{h}t; \ \textit{for all } t \geq T. \tag{1.33}$$

Take ϵ such that $0 < \epsilon < \frac{1}{2}\beta\hat{h}$. From lemma 1.2.3 and the fact that the Y process satisfies (1.29) as well, we could see that there exists $t_2 > 0$ and $\Omega_T^2 \subset \Omega$ for all $T \geq t_2$ with $P(\Omega_T^2) > 1 - \exp(-\delta_2 T)$ for a constant $\delta_2 > 0$, and if $\omega \in \Omega_T^2$, then

$$e^{-\epsilon s} \leq u(s, \gamma s) \ \ for\ all\ s \geq T, \tag{1.34}$$

and

$$e^{-\epsilon s} \leq Y_s \leq e^{\epsilon s} \ \ for\ all\ s \geq T. \tag{1.35}$$

Also, it is easy to see that for any $\lambda > 0$, there exists $t_3 > 0$ such that

$$\exp\left(-\frac{\left(h-\gamma\hat{h}\right)^2}{2D\hat{h}}s\right) + \exp\left(-\frac{\left(h-\frac{1}{2}\gamma\right)^2}{2D}s\right) \leq \frac{1}{8}\lambda \ \ for\ all\ s > t_3. \tag{1.36}$$

Define $t_0 = \max\{t_1, 2t_2, t_3\}$ and $\Omega_T = \Omega_T^1 \cap \Omega_{T/2}^2$ for all $T \geq t_0$. Then it is evident that there exists constant $l > 0$ such that $P(\Omega_T) > 1 - \exp(-lT^{1-p})$, and if $\omega \in \Omega_T$, we have (1.33), (1.34), (1.35) and (1.36) (note in (1.33) and (1.34), the inequalities are true for all $s \geq T/2$). Now, for $T \geq t_0$ fixed, we will use proof by contradiction to show that for any $\omega \in \Omega_T$,

$$\inf_{x<(\gamma-h)t} u(t,x) \geq (1-\lambda)\, Y_t \ \ for\ all\ t > T. \tag{1.37}$$

Suppose that (1.37) is false. This means that for some $\omega \in \Omega_T$, $t \geq T$ and $x^* = x^*(t) < (\gamma - h)t$, it is true that $u(t, x^*) < (1-\lambda)\, Y_t$. Since $u(t,\cdot)$ is a decreasing function, then

$$0 \leq u(t, (\gamma-h)t) \leq u(t, x^*) \leq (1-\lambda)Y_t. \tag{1.38}$$

Let

$$X_r^t = \left(t - r, (\gamma-h)t + \sqrt{D}B_r\right),$$

where B is a standard Brownian motion defined on a probability space $\left(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P}\right)$, which is independent of the Brownian motion W. Let

$$F = \left\{(s,x) : s \geq 0,\ x < \gamma s,\ u(s,x) \leq (1-\frac{\lambda}{2})Y_s\right\}.$$

Then we could see that $X_0^t \in F$, since $t > 0$, $(\gamma - h)t < \gamma t$ and from (1.38), $u(t, (\gamma-h)t) \leq (1-\frac{\lambda}{2})Y_t$. Then we define a stopping time

$$\tau^t = \inf\left\{r \geq 0 : X_r^t \notin F\right\}.$$

Note that $\tau^t < t$. We could check this by checking that $X_t^t \notin F$. The last statement is true since for $x < \gamma(t-t) = 0$, $u(0,x) = Y_0 > (1-\frac{\lambda}{2})Y_0$.

X_r^t may exit F either from the boundary $x = \gamma s$, which we denote by ∂F_1 or the remaining part of boundary of F, which we denote by ∂F_2. Let $\hat{h} \in (0, \frac{h}{\gamma})$ and define

$$
\begin{aligned}
\hat{\Omega}_1 &= \left\{\hat{\omega} \in \hat{\Omega} : X_{\tau^t}^t \in \partial F_1,\ \tau^t \notin \left[\hat{h}t, \frac{1}{2}t\right]\right\}, \\
\hat{\Omega}_2 &= \left\{\hat{\omega} \in \hat{\Omega} : X_{\tau^t}^t \in \partial F_1,\ \tau^t \in \left[\hat{h}t, \frac{1}{2}t\right]\right\}, \\
\hat{\Omega}_3 &= \left\{\hat{\omega} \in \hat{\Omega} : X_{\tau^t}^t \in \partial F_2\right\}.
\end{aligned}
$$

From the Feynman-Kac formula in Lemma 1.2.1 and the strong Markov property for Brownian motion we may conclude that

$$
\begin{aligned}
& u(t,(\gamma-h)t) \\
= \;& \hat{E}\{u(t-\tau^t, (\gamma-h)t + \sqrt{D}B_{\tau^t}) \\
& \exp[\int_0^{\tau^t} c\left(u(t-r,(\gamma-h)t+\sqrt{D}B_r)\right) dr - \frac{\tau^t}{2} + W_t - W_{t-\tau^t}]\} \\
= \;& \hat{E}\left\{u(X_{\tau^t}^t)\exp\left[\int_0^{\tau^t} c\left(u(X_r^t)\right) dr - \frac{\tau^t}{2} + W_t - W_{t-\tau^t}\right]\right\} \\
= \;& \sum_{i=1}^{3} u_i\left(t,(\gamma-h)t\right),
\end{aligned}
$$

where for $i = 1,2,3$

$$
u_i\left(t,(\gamma-h)t\right) = \hat{E}\left\{\mathbf{1}_{\hat{\Omega}_i} u(X_{\tau^t}^t)\exp\left[\int_0^{\tau^t} c\left(u(X_r^t)\right) dr - \frac{\tau^t}{2} + W_t - W_{t-\tau^t}\right]\right\}.
$$

Now we try to obtain a lower bound for $u_2(t,(\gamma-h)t)$. Since we assumed that for $(r,x) \in F$, $u(r,x) \leq (1-\frac{\lambda}{2})Y_r$ and c is a decreasing function, then we have

$$
\int_0^{\tau^t} c\left(u(X_r^t)\right) dr \geq \int_0^{\tau^t} c\left(\left(1-\frac{\lambda}{2}\right)Y_{t-r}\right) dr.
$$

Also note that Y process satisfies (1.29) with initial condition $u_0 \equiv 1$. Therefore we could use the Feynman-kac formula and strong Markov property for Brownian motion to get

$$Y_t Y_{t-\tau^t}^{-1} = \exp\left(\int_0^{\tau^t} c\left(Y_{t-r}\right) dr - \frac{\tau^t}{2} + W_t - W_{t-\tau^t}\right).$$

So we obtain

$$\begin{aligned}
& u_2\left(t, (\gamma - h)t\right) \\
\geq \;& \hat{E}\left\{\mathbf{1}_{\hat{\Omega}_2} u(X_{\tau^t}^t) \exp\left[\int_0^{\tau^t} c\left(\left(1-\frac{\lambda}{2}\right) Y_{t-r}\right) dr - \frac{\tau^t}{2} + W_t - W_{t-\tau^t}\right]\right\} \\
= \;& \hat{E}\left\{\mathbf{1}_{\hat{\Omega}_2} u(X_{\tau^t}^t) Y_t Y_{t-\tau^t}^{-1} \exp\left[\int_0^{\tau^t} c\left(\left(1-\frac{\lambda}{2}\right) Y_{t-r}\right) - c\left(Y_{t-r}\right) dr\right]\right\}.
\end{aligned}$$

Recall definition of $\hat{\Omega}_2$, we know that in $\hat{\Omega}_2$, $\tau^t < t/2$ and hence $t - \tau^t \geq t/2 \geq T/2$. Recall (1.34) and (1.35), we have

$$u\left(t-\tau^t, \left(t-\tau^t\right)\gamma\right) \geq e^{-\epsilon\left(t-\tau^t\right)} > e^{-\epsilon t},$$

and

$$Y_{t-\tau^t}^{-1} \geq e^{-\epsilon\left(t-\tau^t\right)} > e^{-\epsilon t}.$$

Recall (1.33), we have that

$$\int_0^{\tau^t} c\left(\left(1-\frac{\lambda}{2}\right) Y_{t-r}\right) - c\left(Y_{t-r}\right) dr = \int_{t-\tau^t}^{t} c\left(\left(1-\frac{\lambda}{2}\right) Y_r\right) - c\left(Y_r\right) dr > \beta \hat{h} t.$$

Combine the above three inequalities, we could see that

$$u_2\left(t, (\gamma - h)\, t\right) \geq \hat{E}\left\{\mathbf{1}_{\hat{\Omega}_2} e^{-2\epsilon t + \beta \hat{h} t} Y_t\right\} > \hat{P}\left(\hat{\Omega}_2\right) Y_t. \tag{1.39}$$

Consider $u_3\left(t,\left(\gamma-h\right)t\right)$, the same as $u_2\left(t,\left(\gamma-h\right)t\right)$ we can see that

$$
\begin{aligned}
& u_3\left(t,\left(\gamma-h\right)t\right) \qquad (1.40)\\
\geq\ & \hat{E}\left\{\mathbf{1}_{\hat{\Omega}_3} u(X^t_{\tau^t}) Y_t Y^{-1}_{t-\tau^t} \exp\left[\int_0^{\tau^t} c\left(\left(1-\frac{\lambda}{2}\right)Y_{t-r}\right) - c\left(Y_{t-r}\right)dr\right]\right\}\\
\geq\ & \hat{E}\left\{\mathbf{1}_{\hat{\Omega}_3}\left(1-\frac{\lambda}{2}\right) Y_{t-\tau^t} Y_t Y^{-1}_{t-\tau^t} e^{\beta\hat{h}t}\right\}\\
\geq\ & \hat{P}\left(\hat{\Omega}_3\right)\left(1-\frac{\lambda}{2}\right)Y_t.
\end{aligned}
$$

If $\hat{\omega}\in\hat{\Omega}_1$ and $\tau^t<\hat{h}t$, then X^t_s has to meet the line $x=\gamma s$ at a time $\tau^t<\hat{h}t$. This means that

$$
\left(\gamma-h\right)t+\sqrt{D}B_{\tau^t}=\gamma\left(t-\tau^t\right)>\gamma t\left(1-\hat{h}\right),
$$

which is followed by

$$
\sqrt{D}B_{\tau^t}>t\left(h-\gamma\hat{h}\right)>0.
$$

Therefore, by Doob's inequality, we could see that

$$
\begin{aligned}
& \hat{P}\left(\hat{\omega}\in\hat{\Omega}_1,\ and\ \tau^t<\hat{h}t\right)\\
\leq\ & \hat{P}\left(\sqrt{D}B_{\tau^t}\geq t\left(h-\gamma\hat{h}\right),\ for\ \tau^t\in\left[0,\hat{h}t\right]\right)\\
\leq\ & \hat{P}\left(\sup_{0\leq s\leq\hat{h}t}\sqrt{D}B_s\geq t\left(h-\gamma\hat{h}\right)\right)\\
\leq\ & 2\exp\left(-\frac{t\left(h-\gamma\hat{h}\right)^2}{2\hat{h}D}\right).
\end{aligned}
$$

Similarly, if $\hat{\omega}\in\hat{\Omega}_1$ and $\tau^t>t/2$, then X^t_s has to meet the line $x=\gamma s$ at a time $\tau^t>t/2$. This means that

$$
\left(\gamma-h\right)t+\sqrt{D}B_{\tau^t}=\gamma\left(t-\tau^t\right)<\gamma\left(t-t/2\right),
$$

which is followed by

$$\sqrt{D}B_{\tau^t} < t\left(h-\gamma/2\right) < 0.$$

Then, by Doob's inequality,

$$\begin{aligned}
& \hat{P}\left(\hat{\omega}\in\hat{\Omega}_1,\ and\ \tau^t > t/2\right) \\
\leq\ & \hat{P}\left(\sqrt{D}B_{\tau^t} \leq t\left(h-\gamma/2\right),\ for\ \tau^t\in[t/2,t]\right) \\
\leq\ & \hat{P}\left(\inf_{0\leq s\leq t}\sqrt{D}B_s \leq t\left(h-\gamma/2\right)\right) \\
\leq\ & 2\exp\left(-\frac{t\left(h-\gamma/2\right)^2}{2D}\right).
\end{aligned}$$

Recall (1.36), we can see that $\hat{P}\left(\hat{\Omega}_1\right) < \frac{\lambda}{4}$ for all $t\geq T$. Combine the lower bound for $u_2\left(t,\left(\gamma-h\right)t\right)$ and $u_3\left(t,\left(\gamma-h\right)t\right)$ which are (1.39) and (1.40), we can see that for $\omega\in\Omega_T$ and $t\geq T$

$$\begin{aligned}
u\left(t,\left(\gamma-h\right)t\right) &= \sum_{i=1}^{3}u_i\left(t,\left(\gamma-h\right)t\right) \\
&\geq \left(1-\frac{\lambda}{2}\right)Y_t\left(\hat{P}\left(\hat{\Omega}_2\right)+\hat{P}\left(\hat{\Omega}_3\right)\right) \\
&\geq \left(1-\frac{\lambda}{2}\right)Y_t\left(1-\frac{\lambda}{4}\right) \\
&> \left(1-\frac{3}{4}\lambda\right)Y_t.
\end{aligned}$$

for $T > t_0$. This contradicts (1.38). Therefore, we have proved that for any $h\in(0,\gamma/2)$, for any $\lambda>0$, there exists t_0 and Ω_T for all $T\geq t_0$ with $P\left(\Omega_T\right) > 1-\exp(-lT^{1-p})$ for a constant $l>0$, and for any $\omega\in\Omega_T$,

$$\inf_{x<(\gamma-h)t}u\left(t,x\right)\geq\left(1-\lambda\right)Y_t\ for\ all\ t\geq T.$$

□

Now we could use the exact same method as used in proving Theorem 3.4 in [4] to prove the following theorem, which here we just state without proof.

Theorem 1.2.5. *Suppose that function c satisfies condition C1-C4 and u satisfies* (1.29). *Let $\gamma = \sqrt{D\,(2c(0)-1)}$ be the wave speed. Let h be an arbitrary number in $(0,\gamma/2)$ and fix $p \in (0,1)$. For any $\epsilon > 0$, there exist $t_0 > 0$ and $l > 0$, such that for all $T \geq t_0$*

$$P\left\{\frac{1}{t}\int_0^t \inf_{x<s(\gamma-h)} u(s,x)ds > \frac{1}{a}\left(c(0)-\frac{1}{2}\right) - \epsilon \;\; for\ all\ t \geq T\right\} > 1-\exp\left(-lT^{1-p}\right),$$

In particular, almost surely,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \inf_{x<s(\gamma-h)} u(s,x)ds \geq \frac{1}{a}\left(c(0)-\frac{1}{2}\right).$$

Chapter 2

Introduction to Stochastic Partial Differential Equations

The aim of this chapter is to briefly review the basics of stochastic partial differential equations(SPDEs) driven by space time white noise, which will be the content of next few chapters.

First we define the space time white noise we are going to use in those SPDEs we are going to consider, which informally is the derivative of the so called Brownian sheet. Consider the generalized Gaussian zero mean random variable

$$\{W(B) : B \in \mathscr{B}([0,\infty) \times [0,L])\},$$

where $\mathscr{B}([0,\infty) \times [0,L])$is the set of Borel subset of $[0,\infty) \times [0,L]$, defined on $(\Omega, \mathscr{F}, P)$, a complete probability space. If its covariance function is given by

$$E[W(A)W(B)] = \mu(A \cap B),$$

where μ is the Lebesgue measure on $[0,\infty) \times [0,L]$, then the continuous random field $\{W_{tx} := W([0,t] \times [0,x]);\ (t,x) \in [0,\infty) \times [0,L]\}$ is called Brownian sheet.

Similar to the case of Brownian motion, one could define an Ito integral with respect to Brownian sheet. Let $\mathscr{F}_t$ denote the completion of $\sigma(W(B) : B \in \mathscr{B}([0,t] \times [0,L]))$. For any $\{\mathscr{F}_t\}$ predictable function $\phi(\omega, t, x)$:

$\Omega \times [0, \infty) \times [0, L] \to R$ satisfying

$$E\left[\int_0^t \int_0^L \phi^2(\omega, s, x) dx ds\right] < \infty, \ for\ every\ t > 0,$$

one can define a stochastic integral $\int_0^t \int_0^L \phi(\omega, s, x) W(dxds)$ as a continuous $\{\mathscr{F}_t\}$ local martingale whose quadratic variation is given by $\int_0^t \int_0^L \phi^2(\omega, s, x) dx ds$. For more details, see [15].

Next we consider the following class of equations:

$$\begin{aligned} \partial_t u(t,x) &= \triangle u(t,x) + f(u)(t,x) + g(u)(t,x)\ddot{W}_{tx}, \\ u(0,x) &= u_0(x), \ \ 0 \le x \le L; \end{aligned} \tag{2.1}$$

where $\triangle = \frac{\partial^2}{\partial x^2}$, $f(u)(t,x) = f(t,x;u(t,x))$ and $g(u)(t,x) = g(t,x;u(t,x))$. Also, (2.1) is considered with certain boundary conditions, normally one of the following three kinds:

- Periodic boundary condition, i.e. $u(t,0) = u(t,L)$, and $u_x(t,0) = u_x(t,L)$ for all $t \ge 0$;
- Dirichlet boundary condition, i.e. $u(t,0) = u(t,L) = 0$ for all $t \ge 0$;
- Neumann boundary condition, i.e. $u_x(t,0) = u_x(t,L) = 0$ for all $t \ge 0$.

Also, the initial condition $u_0(x)$ satisfies the boundary condition and normally is assumed continuous. It is easy to see that (2.1) is only a notation as the white noise is so rough that any candidate for a solution won't be twice differentiable in x. Therefore, we need a rigorous meaning of this equation. We say that $u(t,x)$ is a solution to (2.1) up to time $T > 0$ if it is locally integrable, adapted to the filtration $\mathscr{F}_t$ which is the completion of $\sigma\left(W(B) : B \in \mathscr{B}\left([0,t] \times [0,L]\right)\right)$, and for any $\phi \in C^2([0,L])$ that satisfies the boundary condition that (2.1) satisfies, it is true that

$$\begin{aligned}
&\int_0^L u(t,x)\phi(x)dx \qquad (2.2)\\
= &\int_0^L u_0(x)\phi(x)dx + \int_0^t\int_0^L \triangle\phi(x)u(s,x)dxds\\
&+\int_0^t\int_0^L \phi(x)f(u)(s,x)dxds + \int_0^t\int_0^L \phi(x)g(u)(s,x)W(dxds),
\end{aligned}$$

where the stochastic integral exists. For all $t \in [0,T]$. One can show that (see [15]) this definition can be extended to test function $\psi(t,x)$ so that for $\psi(t,x) \in C^{1,2}([0,T]\times[0,L])$ and at each $t \in [0,T]$ satisfies the boundary condition that (2.1) satisfies, if $u(t,x)$ is a solution, then

$$\begin{aligned}
&\int_0^L u(t,x)\psi(t,x)dx \qquad (2.3)\\
= &\int_0^L u_0(x)\psi(0,x)dx + \int_0^t\int_0^L u(s,x)\left[\triangle\psi(x) + \partial_t\psi(t,x)\right]dxds\\
&+\int_0^t\int_0^L \psi(s,x)f(u)(s,x)dxds + \int_0^t\int_0^L \psi(s,x)g(u)(s,x)W(dxds).
\end{aligned}$$

for all $t \in [0,T]$. There is another way to write equation (2.1) in an integrated form, using G, the Green's function of the one dimensional Laplacian operator with corresponding boundary conditions. It was proved in [15] that $u(t,x)$ is a solution to (2.1) up to time T if at any point $(t,x) \in [0,T]\times[0,L]$,

$$\begin{aligned}
u(t,x) \;=\; &\int_0^L G_t(x,y)u_0(y)dy + \int_0^t\int_0^L G_{t-s}(x,y)f(u)(s,y)dyds (2.4)\\
&\int_0^t\int_0^L G_{t-s}(x,y)g(u)(s,y)W(dxds).
\end{aligned}$$

Also in [15], it was proved that when u_0 is Hölder continuous, f and g are jointly measurable, Lipschitz with respect to their third arguement uniformly in (t,x), and that for any $T>0$,

$$\int_0^T\int_0^L \left[f^2(t,x;0) + g^2(t,x;0)\right]dxdt < \infty,$$

then the equation (2.1) has a unique strong solution which is Hölder $1/2-\epsilon$ continuous in t, Hölder $1/4-\epsilon$ continuous in x, for any $\epsilon > 0$, and is adapted to the filtration generated by the Brownian sheet. There are many results on existence and uniqueness of solution to (2.1) and its regularity under different assumptions for the coefficients f and g, and the initial condition u_0, see e.g. [35], [11], [12], [10], [13], [21], [34], [32].

As a special case, we consider equation (2.1) for $g(u)(t,x) = \sqrt{u(t,x)}$, and $f(u)(t,x) = f(u(t,x))$, such that f is Lipschitz and that $f(0) \geq 0$, and initial condition $u_0(x)$ is nonnegative and Hölder continuous to the order $1/4$. This equation arises naturally as a continuum limit of certain interacting particle systems (see [32]) In [34], the author considered this equation on domain R, not on $[0,L]$. He proved that for any continuous initial condition which is growing slower than exponentially, there exists a weak solution which is unique in law, almost surely nonnegative and continuous. It is possible to use the same method to prove these same results for the solution of (2.1) on $[0,L]$ with the above boundary conditions. Next, we note that when $f(u) \equiv 0$, equation (2.1) becomes

$$\begin{aligned} \partial_t u(t,x) &= \triangle u(t,x) + \sqrt{u(t,x)}\ddot{W}_{tx}, \\ u(0,x) &= u_0(x), \quad 0 \leq x \leq L. \end{aligned} \tag{2.5}$$

Here we consider this equation with certain boundary condition, periodic, Dirichlet or Neumann. Then $X_t(dx) = u(t,x)dx$ defines a measure-valued branching diffusion process with branching rate $1/2$, diffusion constant 1 and initial measure $u_0(x)dx$ on $[0,L]$ with certain boundary condition (see [28]). By measure-valued branching diffusion process with branching rate ρ, diffusion constant κ and initial measure μ, we mean that the continuous random process $t \to X_t$ on time interval $[0,T]$, $T > 0$, with values in a set $\mathfrak{M}$ which is a set of measures on $[0,L]$ endowed with the topology of weak convergence, and distribution $P_\mu = P_\mu^{\rho,\kappa}$ as the solution to the following martingale problem:

- At initial time $t = 0$, $P_\mu[X_0 = \mu] = 1$;
- For each sufficiently regular map $t \to f_t$, with $f_t \in C^2([0,L])$ satisfying the boundary condition for all $t \in [0,T]$, and $C^2([0,L])$ is endowed

with the supreme norm topology, there is a continuous mean zero P_μ martingale $M_t = M(f)$, $t \in [0,T]$, such that P_μ a.s.

$$\langle X_t, f_t \rangle = \langle \mu, f_0 \rangle + \int_0^t \langle X_s, \partial_s f_s + \kappa \triangle f_s \rangle \, ds + M_t, \; t \in [0,T],$$

and with quadratic variation $[M]$ given by

$$[M]_t = 2\rho \int_0^t \langle X_s, f_s^2 \rangle \, ds, \; t \in [0,T], a.s.$$

Here $\langle \nu, h \rangle = \int_0^L h(x)\nu(dx)$, for any measure ν on $[0,L]$ and any function h. We could view this martingale problem as the corresponding version of equation (2.5) using the language of measure-valued branching diffusion process. We will use this idea, i.e. the solution of (2.5) and the solution to the martingale problem are related to explore some large deviation results in the next chapter.

Before we end this short introduction, we state and briefly prove a little lemma which will be used later in Chapter 4. Suppose that $u(t,x)$ is the solution to the equation (2.5) considered with Neumann boundary condition. Note that through (2.3), we would know the representation for $\int_0^T \int_0^L \psi(t,x)\sqrt{u(t,x)}W(dxdt)$ for any smooth function $\psi(t,x)$ which satisfies Neumann boundary condition for all $t \in [0,T]$. We would like to derive similar representation formula for $\int_0^T \int_0^L \theta(t,x)\sqrt{u(t,x)}W(dxdt)$ for those smooth $\theta(t,x)$ which may not satisfy Neumann boundary condition. We claim the following lemma is true.

Lemma. *Suppose $u(t,x)$ satisfies equation (2.5) with Neumann boundary conditions. Then for any smooth function $\theta(t,x)$ defined on $[0,T] \times [0,L]$,*

$$\begin{aligned} & \int_0^T \int_0^L \theta(t,x)\sqrt{u(t,x)}W(dxdt) \qquad (2.6) \\ = & \int_0^L [\theta(T,x)u(T,x) - \theta(0,x)u_0(x)] \, dx + \int_0^T [u(t,L)\theta_x(t,L) - u(t,0)\theta_x(t,0)] \, dt \\ & - \int_0^T \int_0^L u(t,x) \left(\triangle\theta(t,x) + \theta_t(t,x) \right) dxdt. \end{aligned}$$

Proof. We are going to prove the following special case which will be the key building block to prove the whole Lemma. We claim that if $u(t,x)$ is the solution to (2.5), and $\vartheta(x)$ is a smooth function defined on $[0,L]$ such that $\vartheta_x(0)\neq 0$, but $\vartheta_x(L)=0$, then

$$\begin{aligned}
&\int_0^T\int_0^L \vartheta(x)\sqrt{u(t,x)}W(dxdt) \qquad (2.7)\\
=\;& \int_0^L \vartheta(x)\left[u(T,x)-u(0,x)\right]dx-\int_0^T \vartheta_x(0)u(t,0)dt-\int_0^T\int_0^L \triangle\vartheta(x)u(t,x)dxdt.
\end{aligned}$$

To see that this claim is true, we define a function $\eta(x)$ on $[0,L]$ such that it is smooth, compactly supported inside $(0,L)$ and satisfies $\int_0^L \eta(x)dx=1$. Then for any $\epsilon>0$, define $\eta_\epsilon(x)=\frac{1}{\epsilon}\eta(\frac{x}{\epsilon})$, and $H_\epsilon(x)=\int_x^L\int_y^L \eta_\epsilon(z)dzdy$. Then through simple calculation we could see that the function $H_\epsilon(x)$ satisfies

$$\triangle H_\epsilon(x)=\eta_\epsilon(x),\;\; H_\epsilon(x)=0,\; for\; x\geq\epsilon,\;\; H_\epsilon^{'}(0)=-1,\;\; 0\leq H_\epsilon(x)\leq\epsilon. \qquad (2.8)$$

So, $\vartheta(x)+\vartheta_x(0)H_\epsilon(x)$ is smooth and satisfies the Neumann boundary condition at both 0 and L. Therefore we could use (2.2) to see that

$$\begin{aligned}
&\int_0^T\int_0^L \vartheta(x)\sqrt{u(t,x)}W(dxdt) \qquad (2.9)\\
=\;& -\int_0^T\int_0^L \vartheta_x(0)H_\epsilon(x)\sqrt{u(t,x)}W(dxdt)\\
&+\int_0^L (u(T,x)-u(0,x))\left(\vartheta(x)+\vartheta_x(0)H_\epsilon(x)\right)dx\\
&-\int_0^T\int_0^L u(t,x)\triangle\left(\vartheta(x)+\vartheta_x(0)H_\epsilon(x)\right)dxdt
\end{aligned}$$

Now let $\epsilon\to 0$, since for almost all fixed ω, $(t,x)\to u(t,x)$ is continuous, using (2.8) it is not difficult to see that for almost all ω at any fixed $t\in[0,T]$,

$$\lim_{\epsilon\to 0}\int_0^T\int_0^L u(t,x)\vartheta_x(0)\eta_\epsilon(x)dxdt=\int_0^T u(t,0)\vartheta_x(0)dt,$$

$$\lim_{\epsilon\to 0}\int_0^T\int_0^L u^2(t,x)\left(\vartheta_x(0)H_\epsilon(x)\right)^2dxdt=0,$$

$$\lim_{\epsilon \to 0} \int_0^L \left(u(T,x) - u(0,x)\right) \vartheta_x(0) H_\epsilon(x) dx = 0.$$

Therefore (2.7) is true. A similar procedure allows one to handle the other end point at $x = L$ and finally we could follow standard method (see section II of [6]) to add the dependence of t to function θ and derive (2.6). $\square$

Remark 2.0.6. When u solves (2.5) with Dirichlet Boundary conditions, the extra term should be informally $\int_0^T u_x(t,L)\theta(t,L) - u_x(0,L)\theta(0,L)dt$, which since u is not differentiable at 0 nor L, needs some interpretation.

Chapter 3

Large Deviations for Super-Brownian Motions

Consider the one dimensional stochastic partial differential equation(SPDE)

$$\begin{aligned} \partial_t u^\epsilon(t,x) &= \partial_{xx}u^\epsilon(t,x) + \epsilon\sqrt{u^\epsilon(t,x)}\ddot{W}_{tx}, \\ u^\epsilon(0,x) &= \zeta(x) \end{aligned} \tag{3.1}$$

where W is a time-space white noise on a probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ based on domain $[0,T]\times[0,L]$, $\zeta(x)\geq 0$ is a continuous function on $[0,L]$. Also we consider this SPDE with either Dirichlet or Neumann boundary conditions.

As $\epsilon \to 0$, we expect the solution of (3.1) will tend to the solution of heat equation in some way. Our aim is to establish a large deviation result for u^ϵ as a random perturbation of the solution of heat equation. i.e. We would like to find a rate function $I(\phi)$ in Large Deviation Principle(LDP), which roughly says that

$$\lim_{\epsilon\to 0}\epsilon^2 \log P(u^\epsilon \in A) = -\inf_{\phi\in A} I(\phi)$$

where A is a subset of some space in which the solution of (3.1) is considered.

For the one dimensional SPDE (3.1), we could consider its solution in the following two spaces:

- The space of nonnegative continuous functions $u^\epsilon : [0,T]\times[0,L] \to R$, with initial condition $u^\epsilon(0,x) = \zeta(x)$. We denote this space by

$C_\zeta^+([0,T]\times[0,L])$. i.e. The solution is considered as a real valued random field indexed by t and x.

- The space of continuous maps $X^\epsilon : [0,T] \to \mathfrak{M}$, where $\mathfrak{M}$ is the space of all finite measures on $[0,L]$ equipped with the weak topology, with initial measure $X_0^\epsilon = \mu$, where $\mu(A) = \int_A \zeta(x)dx$, for Borel subsets A of $[0,L]$. We denote this space by $\mathcal{C}_\mu([0,T],\mathfrak{M})$. i.e. The solution is considered as a stochastic process taking values in $\mathfrak{M}$ indexed by t.

The connection between these two spaces can be easily seen by letting the process X_t^ϵ be, for each $t \in [0,T]$

$$X_t^\epsilon(A) = \int_A u^\epsilon(t,x)dx. \tag{3.2}$$

for any Borel subset A of $[0,L]$.

3.1 Review of LDP when the solution is considered in $\mathcal{C}_\mu([0,T],\mathfrak{M})$

In [20], the authors proved a LDP result for Super-Brownian motion, which is the solution of a martingale problem. By posing (3.1) in its weak form, we could see the equivalence between Super-Brownian motion and the solution of SPDE (3.1) when it is considered in $\mathcal{C}_\mu([0,T],\mathfrak{M})$.

Let $(\theta(t,x) : (t,x) \in [0,T]\times[0,L])$, be a smooth function satisfying the same boundary condition as (3.1). Then (3.1) implies that for all such $\theta(t,x)$, we have

$$\begin{aligned} &\int_0^L [u^\epsilon(t,x)\theta(t,x) - u^\epsilon(0,x)\theta(0,x)]dx \\ = &\int_0^t\int_0^L u^\epsilon(s,x)[\partial_{xx}\theta(s,x) + \partial_t\theta(s,x)]dxds \\ &+ \int_0^t\int_0^L \epsilon\sqrt{u^\epsilon(s,x)}\theta(s,x)W(dxds). \end{aligned} \tag{3.3}$$

Let X_t^ϵ be the process defined as in (3.2). Note that

$$M_t := \int_0^t\int_0^L \epsilon\sqrt{u^\epsilon(s,x)}\theta(s,x)W(dxds)$$

is a martingale with respect to $\mathscr{F}_t$. Then the weak form (3.3) becomes a martingale problem and we can see that X_t^ϵ is the super-Brownian motion as defined in section 0.1 of [20].

In [20], the authors proved that $\{P^\epsilon\}$ which are probability measures on $\mathcal{C}_\mu([0,T],\mathfrak{M})$ (whose topology will be specified below) induced by $X_\cdot^\epsilon$ satisfy large deviation principles with a good rate function(to be specified below). Here is a quick review of their result.

First we review the topology they used in the theorem. Let Φ denote the space of continuous functions $\varphi : [0,L] \to R$. Let Φ^* denote the dual space of Φ and endow it with the topology of weak convergence generated by the Prohorov metric ρ_0. Then $\mathfrak{M}$, the space of all finite measures on $[0,L]$, can be viewed as a subset of Φ^*. So it is equipped with the subspace topology. Then $\mathcal{C}_\mu([0,T],\mathfrak{M})$ is equipped with the compact-open topology, i.e. the topology of the convergence defined as follows, for any sequence $\{f_n(\cdot)\} \in \mathcal{C}_\mu([0,T],\mathfrak{M})$, $f_n \to f$ if $\sup_{t\leq T} |f_n(t) - f(t)|_{\rho_0} \to 0$.

Next we describe the rate function. The space of test functions, denoted by $\Phi^{2+\gamma}$, is the space of $(2+\gamma)$-Hölder continuous functions $\varphi : [0,L] \to R$ that satisfy the same boundary condition as the SPDE (3.1), for some $0 < \gamma < 1$. For any $\mu \in \mathfrak{M}$, let $\langle \mu, \varphi \rangle$ stand for $\int_0^L \varphi(x)\mu(dx)$. Let $\Phi^{2+\gamma}_{[0,L]}$ be the Hölder space $\mathcal{C}^{2+\gamma}([0,T]\times[0,L])$ which additionally satisfies the same boundary condition as the SPDE (3.1) for all times $t \in [0,T]$. Let $(\Phi^{2+\gamma})^*$ be the dual space of $\Phi^{2+\gamma}$. Since $\mathfrak{M}$ can be considered as a topological subset of $(\Phi^{2+\gamma})^*$, we extend the notation $\varphi \to \langle \vartheta, \varphi \rangle$ used for $\vartheta = \mu \in \mathfrak{M}$ to any $\vartheta \in (\Phi^{2+\gamma})^*$. Consider the Laplacian Δ as an operator on $\Phi^{2+\gamma}$. Its dual operator Δ^* is defined on $(\Phi^{2+\gamma})^*$ by

$$\langle \Delta^*\vartheta, \varphi \rangle = \langle \vartheta, \Delta\varphi \rangle,\ \vartheta \in (\Phi^{2+\gamma})^*,\ \varphi \in \Phi^{2+\gamma}.$$

Both of the operators depend on the choice of boundary conditions. A map $t \to \vartheta_t \in (\Phi^{2+\gamma})^*$ defined for $t \in [0,T]$ is said to be absolutely continuous if there is an absolutely continuous real-valued function k on $[0,T]$, such that

$$|\langle \vartheta_t, \varphi \rangle - \langle \vartheta_s, \varphi \rangle| \leq |k(t) - k(s)|,\ s,t \in [0,T],\ \varphi \in \Phi^{2+\gamma},\ \|\varphi\|_{2+\gamma} \leq 1$$

An absolutely continuous map possesses a time derivative $\frac{d}{dt}\vartheta_t = \dot{\vartheta}_t \in (\Phi^{2+\gamma})^*$ in the distribution sense for almost all $t \in [0,T]$ and the integration by parts formula holds for all $f \in \Phi^{2+\gamma}_{[0,L]}$

$$\int_s^t \langle \dot{\vartheta}_r, f_r \rangle dr = \langle \vartheta_t, f_t \rangle - \langle \vartheta_s, f_s \rangle - \int_s^t \langle \vartheta_r, \dot{f}_r \rangle dr, \ 0 \leq s < t \leq T \tag{3.4}$$

For any generalized function $\vartheta \in (\Phi^{2+\gamma})^*$, it is absolutely continuous with respect to a measure $\mu \in \mathfrak{M}$ if there is a nonnegative function g which is μ-integrable and satisfies that $\langle \vartheta, \varphi \rangle = \langle \mu, g\varphi \rangle$, for all $\varphi \in \Phi^{2+\gamma}$. And in this case, $g = \frac{d\vartheta}{d\mu}$, is called the Radon-Nikodym derivative of ϑ with respect to μ. The space H in which the rate function is properly defined and is finite is defined as in Definition 1.4.2 in [20]. For $\nu \in H \subset \mathcal{C}_\mu([0,T], \mathfrak{M})$, it is required that the map $t \to \nu_t \in (\Phi^{2+\gamma})^*$ defined on $[0,T]$ is absolutely continuous, $\dot{\nu}_t - \triangle^* \nu_t \in (\Phi^{2+\gamma})^*$ is absolutely continuous with respect to ν_t for almost all $t \in [0,T]$, and if we denote the Radon-Nikodym derivative by h_t, the map $t \to h_t$ belongs to $L^2(\nu) = L^2([0,T] \times [0,L], dr\nu_r(dy))$, which means

$$\begin{aligned} & \int_0^T \| \frac{d(\dot{\nu}_t - \triangle^* \nu_t)}{d\nu_t} \|^2_{L^2(\nu_t)} dt = \int_0^T \|h_t\|^2_{L^2(\nu_t)} dt \\ = & \int_0^T \int_0^L h_t^2(y) \nu_t(dy) dt \\ = & \int_0^T \langle \nu_t, h_t^2 \rangle dt < \infty \end{aligned}$$

Then the rate function is defined as

$$S(\nu) = \begin{cases} \frac{1}{2} \int_0^T \| \frac{d(\dot{\nu}_t - \triangle^* \nu_t)}{d\nu_t} \|^2_{L^2(\nu_t)} dt & \text{if } \nu \in H \\ \infty & \text{otherwise} \end{cases} \tag{3.5}$$

Finally, in Theorem 1.6.1 [20], they stated a Schilder Type theorem, which says the super-Brownian motion X^ϵ defined as in (3.2) with either Dirichlet or Neumann boundary conditions and living in $\mathcal{C}_\mu([0,T], \mathfrak{M})$ which is equipped with the compact open topology, satisfiy a LDP with a good rate function defined as in (3.5).

3.2 LDP when the solution is considered in $C_\zeta^+([0,T]\times[0,L])$

Given the large deviation principle proved in [20], we would like to prove a large deviation principle result for $\{P^\epsilon,\ \epsilon>0\}$ which are probability measures on $C_\zeta^+([0,T]\times[0,L])$ induced by (3.1), when the space is equipped with sup norm topology and find the corresponding good rate function.

In our proof, we will use the exponential tightness of $\{P^\epsilon,\ \epsilon>0\}$. Also this property is important on its own. So we use the next section to prove it first.

3.2.1 An Exponential Tightness Result

A similar result is proved in Appendix A, Proposition A.2. in [31] by R.Sowers. We will adjust his result to our problem. Basically, we will change the boundary conditions to the ones we are going to use and solve the problem that in our case, the noise coefficient term is not a bounded function.

First let us recall that the family $\{P^\epsilon, \epsilon>0\}$ of measures on a Polish space $\mathfrak{X}$ with Borel sigma-field $\mathscr{B}(\mathfrak{X})$ is said to be exponentially tight if for each $M>0$, there is a compact subset K_M of $\mathfrak{X}$ such that

$$\limsup_{\epsilon\to0}\epsilon^2\ln P^\epsilon(K_M^c)\le -M.$$

See [16]. Normally, we use the equivalent definition that for each $M>0$, there is a compact subset K_M of $\mathfrak{X}$ and

$$\limsup_{\epsilon\to0}\epsilon^2\ln P^\epsilon(K_M^c)\le -l(M),$$

for some function $l(M)$ satisfying $\lim_{M\to\infty} l(M)=\infty$.

In our problem, the space is $C_\zeta^+([0,T]\times[0,L])$. For some $0<\kappa<\frac{1}{4}$, define the κ-th order Hölder norm $\|\cdot\|_\kappa$ as

$$\|\phi\|_\kappa=\sup_{(t,x)\in[0,T]\times[0,L]}|\phi(t,x)|+\sup_{(t,x)(s,y)\in[0,T]\times[0,L],(t,x)\neq(s,y)}\frac{|\phi(t,x)-\phi(s,y)|}{(r((t,x),(s,y)))^\kappa}.$$

where $r\left((t,x),(s,y)\right)$ is the Euclidean distance between (t,x) and (s,y). When $C_\zeta^+([0,T]\times[0,L])$ is equipped with the sup-norm topology, every closed subset whose κ-th Hölder norm is bounded is a compact subset of $C_\zeta^+([0,T]\times[0,L])$.

We will base our proof on the proof of Proposition A.2 in [31]. There, the author considered the Hölder norm of solutions to the SPDE

$$\begin{aligned}\partial_t \Xi(t,x) &= \triangle \Xi(t,x) + \sigma(\omega,t,x)\ddot{W}_{tx},\ (t,x)\in[0,T]\times S^1, \qquad (3.6)\\ \Xi(0,x) &\equiv 0.\end{aligned}$$

In his setting, the space variable lies in $S^1 = \{e^{i\theta} : \theta \in R\}$ and the SPDE satisfies periodic boundary condition. Also $\sigma(\omega,t,x)$ is in $L^\infty(\Omega\times[0,T]\times S^1)$ such that σ is P-a.s. continuous as a function of (t,x) in $[0,T]\times S^1$ and such that σ is adapted to $\mathscr{F}_t = \sigma\{W(A) : A \in \mathscr{B}([0,t]\times S^1)\}$. Let G denote the Green's function for the Laplacian operator with periodic boundary conditions. He first wrote the solution to (3.6) as

$$\Xi(t,x) = \int_0^t \int_{S^1} G_{t-s}(x,y)\sigma(s,y)W(dsdy). \qquad (3.7)$$

Then he proved the following exponential tightness result using Proposition A.1 in [31], an estimate on the Green's function.

Lemma 3.2.1. *For each $0<\kappa<\frac{1}{4}$, there are positive constants K_κ^0 and K_κ^1 depending only on κ such that*

$$P\{\|\Xi\|_\kappa \geq L\} \leq \exp\{-K_\kappa^1(\frac{L}{\|\sigma\|_\infty})^2\},$$

for all $L>0$ such that $L \geq K_\kappa^0\|\sigma\|_\infty$. Here $\|\sigma\|_\infty = \|\sigma\|_{L^\infty(\Omega\times[0,T]\times S^1)}$.

We will check that this lemma still holds when we consider (3.6) on domain $[0,T]\times[0,L]$, and change the boundary condition to either Dirichlet or Neumann boundary conditions. Examining the details of the proof of this lemma, we could see that the only thing that needs checking is the following lemma about estimates on Green's functions corresponding to different boundary conditions.

Lemma 3.2.2. *Let G be the Green's function for the equation $\partial_t v(t,x) = \triangle v(t,x)$, $(t,x)\in(0,T)\times(0,L)$ with either Dirichlet or Neumann boundary conditions. For $0<\kappa<\frac{1}{4}$, there is a positive number C_κ^1, such that for all*

(t,x) and (s,y) in $[0,T]\times[0,L]$, we have

$$\{\int_0^\infty\int_0^L |G_{t-r}(x,z)-G_{s-r}(y,z)|^2 dzdr\}^{1/2} \le C_\kappa^1 r((t,x),(s,y))^\kappa. \quad (3.8)$$

where $r((t,x),(s,y))$ is the Euclidean distance between (t,x) and (s,y).

Proof. The proof here essentially is similar to the proof of Proposition A.1 in [31]. It is easy to see that

$$\begin{aligned} & \int_0^\infty\int_0^L |G_{t-r}(x,z)-G_{s-r}(y,z)|^2\, dzdr \qquad (3.9)\\ \le\ & 2\int_0^\infty\int_0^L |G_{t-r}(x,z)-G_{t-r}(y,z)|^2\, dzdr + 2\int_0^\infty\int_0^L |G_{t-r}(y,z)-G_{s-r}(y,z)|^2\, dzdr. \end{aligned}$$

Let $\{\phi_n\}$ be the orthonormal basis of $L^2([0,L])$ consisting of the eigenfunctions of Laplacian with either Dirichlet or Neumann boundary conditions and let $\{-\lambda_n\}$ be the corresponding eigenvalues. Then we could write the Green's function as

$$G_{t-r}(x,z) = \mathbf{1}_{\{t-r\ge 0\}} \sum_{n=0}^\infty \exp\left(-\lambda_n\,(t-r)\right)\phi_n(x)\phi_n(z),$$

where G and $\{\phi_n\}$ have the same boundary condition. For $t>0$, $x,y\in[0,L]$, we can see that

$$\begin{aligned} & \int_0^\infty\int_0^L |G_{t-r}(x,z)-G_{t-r}(y,z)|^2\, dzdr\\ =\ & \int_0^\infty\int_0^L \left[\mathbf{1}_{\{t-r\ge 0\}}\sum_{n=0}^\infty \exp\left(-\lambda_n\,(t-r)\right)\phi_n(z)\left(\phi_n(x)-\phi_n(y)\right)\right]^2 dzdr\\ =\ & \sum_{n=0}^\infty \left(\phi_n(x)-\phi_n(y)\right)^2 \int_0^\infty \mathbf{1}_{\{t-r\ge 0\}}\exp\left(-\lambda_n\,(t-r)\right) dr\\ =\ & \sum_{n=0}^\infty \frac{1-\exp\left(-2\lambda_n t\right)}{2\lambda_n}\left(\phi_n(x)-\phi_n(y)\right)^2. \end{aligned}$$

If the Laplacian operator satisfies Dirichlet boundary condition, then

$$\phi_n(x) = \sqrt{\frac{2}{L}}\sin\sqrt{\lambda_n}x = \sqrt{\frac{2}{L}}\sin\frac{n\pi}{L}x,$$

and when the Laplacian operator satisfies Neumann boundary condition, then

$$\phi_n(x) = \sqrt{\frac{2}{L}} \cos \sqrt{\lambda_n} x = \sqrt{\frac{2}{L}} \cos \frac{n\pi}{L} x.$$

In both cases, we can see that obviously, the eigenfunctions satisfy

$$|\phi_n(x) - \phi_n(y)| \leq 2\sqrt{\frac{2}{L}}.$$

Also, from mean value theorem,

$$|\phi_n(x) - \phi_n(y)| \leq \sqrt{\frac{2}{L}} \sqrt{\lambda_n} \left|x - y\right|.$$

We then could see that

$$\begin{aligned} & \int_0^\infty \int_0^L |G_{t-r}(x,z) - G_{t-r}(y,z)|^2 \, dz dr \qquad (3.10) \\ \leq & \sum_{n=0}^{\infty} \frac{1}{2\lambda_n} |\phi_n(x) - \phi_n(y)|^{2\kappa} |\phi_n(x) - \phi_n(y)|^{2-2\kappa} \\ \leq & \sum_{n=0}^{\infty} \frac{1}{2\lambda_n} \left[\sqrt{\frac{2}{L}} \sqrt{\lambda_n} |x - y|\right]^{2\kappa} \left[2\sqrt{\frac{2}{L}}\right]^{2-2\kappa} \\ = & L_1 |x - y|^{2\kappa}. \end{aligned}$$

where $L_1 = 2^{2-2\kappa} L^{-1} \sum_{n=0}^{\infty} \frac{1}{\lambda_n^{1-\kappa}}$. since $2(1-\kappa) > 1$, L_1 is a finite number. Now take $0 \leq s < t$, then

$$\begin{aligned} & \int_0^\infty \int_0^L |G_{t-r}(y,z) - G_{s-r}(y,z)|^2 \, dz dr \\ = & \int_0^\infty \int_0^L \left[\sum_{n=0}^{\infty} \phi_n(y)\phi_n(z) \left(\mathbf{1}_{\{t \geq r\}} e^{-\lambda_n (t-r)} - \mathbf{1}_{\{s \geq r\}} e^{-\lambda_n (s-r)}\right)\right]^2 dz dr \\ \leq & \frac{2}{L} \sum_{n=0}^{\infty} \int_0^\infty \left(\mathbf{1}_{\{t \geq r\}} e^{-\lambda_n (t-r)} - \mathbf{1}_{\{s \geq r\}} e^{-\lambda_n (s-r)}\right)^2 dr. \end{aligned}$$

Here we used the fact that the eigenfunctions corresponding to both boundary conditions satisfy $\phi_n^2(y) \leq \frac{2}{L}$. Then we can see that

$$
\begin{aligned}
& \int_0^\infty \left(\mathbf{1}_{\{t\geq r\}} e^{-\lambda_n(t-r)} - \mathbf{1}_{\{s\geq r\}} e^{-\lambda_n(s-r)}\right)^2 dr \\
= \; & \int_0^s \left(e^{-\lambda_n(t-s+r)} - e^{-\lambda_n r}\right)^2 dr + \int_0^{t-s} e^{-2\lambda_n r} dr \\
= \; & \left(e^{-\lambda_n(t-s)} - 1\right)^2 \frac{1-e^{-2\lambda_n s}}{2\lambda_n} + \int_0^{t-s} e^{-2\lambda_n r} dr \\
\leq \; & \frac{1-e^{-\lambda_n(t-s)}}{2\lambda_n} + \int_0^{t-s} e^{-\lambda_n r} dr \\
= \; & \frac{3}{2} \int_0^{t-s} e^{-\lambda_n r} dr.
\end{aligned}
$$

If we take $\kappa^{'} = \frac{1}{1-2\kappa}$, then $1 < \kappa^{'} < 2$. Then use Jensen's inequality, we have that

$$
\left(\frac{1}{t-s}\int_0^{t-s} e^{-\lambda_n r} dr\right)^{\kappa^{'}} \leq \frac{1}{t-s}\int_0^{t-s} e^{-\lambda_n \kappa^{'} r} dr \leq \frac{1}{t-s}\frac{1}{\lambda_n \kappa^{'}}.
$$

Therefore,

$$
\int_0^{t-s} e^{-\lambda_n r} dr \leq \left\{(t-s)^{k^{'}-1}\frac{1}{\lambda_n \kappa^{'}}\right\}^{1/\kappa^{'}} = (t-s)^{2\kappa}\left(\frac{1}{\lambda_n \kappa^{'}}\right)^{1/\kappa^{'}},
$$

and

$$
\int_0^\infty \int_0^L |G_{t-r}(y,z) - G_{s-r}(y,z)|^2 \, dz dr \leq L_2 \, (t-s)^{2\kappa}. \tag{3.11}
$$

where $L_2 = \frac{3}{L}\sum_{n=0}^\infty \left(\frac{1}{\lambda_n \kappa^{'}}\right)^{1/\kappa^{'}}$. Since $2/\kappa^{'} > 1$, L_2 is a finite number. Now combine (3.9), (3.10) and (3.11), we could see that (3.8) is true by taking $C_\kappa^1 = \sqrt{2L_1 + 2L_2}$.

□

Remark 3.2.3. Now, Lemma 3.2.1 is valid for Ξ satisfies the equation and initial condition in (3.6) but considered on domain $[0,T]\times[0,L]$, with either Dirichlet or Neumann boundary condition.

Now we are ready to state and prove our exponential tightness result.

Lemma 3.2.4. *Suppose that $u^\epsilon(t,x)$ satisfies* (3.1) *with either Dirichlet or Neumann boundary conditions. Suppose that $u_1^\epsilon(t,x)$ satisfies*

$$\begin{aligned} \partial_t u_1^\epsilon(t,x) &= \triangle u_1^\epsilon(t,x) + \epsilon\sqrt{u^\epsilon(t,x)}\ddot{W}_{tx}, \\ u_1^\epsilon(0,x) &= 0, \end{aligned} \tag{3.12}$$

with the same boundary condition as $u^\epsilon(t,x)$ but with initial condition $u_1^\epsilon(0,x) \equiv 0$. For each $0 < \kappa < \frac{1}{4}$, there are positive constants K_κ^0 and K_κ^1 depending only on κ and an M_0 satisfying $M_0 \geq K_\kappa^0 \epsilon\sqrt{M_0 + \|\zeta\|_\infty}$ so that for any $M \geq M_0$, we have

$$P\left\{\|u_1^\epsilon\|_\kappa \geq M\right\} \leq \exp\left\{-\frac{K_\kappa^1}{\epsilon^2}\frac{M^2}{M+\|\zeta\|_\infty}\right\}.$$

Also, the the family of probability measures $\{P^\epsilon, \epsilon > 0\}$ induced by $u^\epsilon(t,x)$ on $C_\zeta^+([0,T]\times[0,L])$ is exponentially tight.

Proof. Let G be the Green's function for the equation $\partial_t v(t,x) = \triangle v(t,x)$, $(t,x) \in [0,T]\times[0,L]$ with the same boundary condition as SPDE (3.1). Then we could represent any solution to (3.1) as

$$\begin{aligned} u^\epsilon(t,x) &= \int_0^L G_t(x,y)\zeta(y)dy + \int_0^t\int_0^L G_{t-s}(x,y)\epsilon\sqrt{u^\epsilon(s,y)}W(dyds) \\ &:= u_2(t,x) + u_1^\epsilon(t,x), \end{aligned}$$

where u_1^ϵ satisfies (3.12) with the same boundary condition as (3.1). Define stopping time

$$\tau_N = \inf\left\{t : \sup_{x\in[0,L]} u^\epsilon(t,x) > N\right\},$$

and construct $v^\epsilon(t,x)$ as

$$v^\epsilon(t,x) = \begin{cases} u_1^\epsilon(t,x) & \text{if } t \leq \tau_N, \\ \int_0^L G_{t-\tau_N}(x,y)u_1^\epsilon(\tau_N,y)dy & \text{if } t > \tau_N. \end{cases}$$

Then, we could see that $v^\epsilon(t,x)$ satisfies

$$\begin{aligned} \partial_t v^\epsilon(t,x) &= \triangle v^\epsilon(t,x) + \epsilon\sqrt{u^\epsilon(t,x)}\mathbf{1}_{(t\leq\tau_N)}\ddot{W}_{tx}, \\ v^\epsilon(0,x) &= 0. \end{aligned} \tag{3.13}$$

Now since $\epsilon\sqrt{u^\epsilon(t,x)}\mathbf{1}_{(t\leq\tau_N)} \leq \epsilon\sqrt{N}$, we could apply Remark 3.2.3 and see that for all $M \geq K_\kappa^0\epsilon\sqrt{N}$,

$$P\left\{\|v^\epsilon\|_\kappa \geq M\right\} \leq \exp\left\{-K_\kappa^1\left(\frac{M}{\epsilon\sqrt{N}}\right)^2\right\}. \tag{3.14}$$

Now consider $u_2(t,x) = \int_0^L G_t(x,y)\zeta(y)dy$, it is easy to see that $\|u_2\|_{\text{sup}} \leq \|\zeta\|_\infty$. Then for any M such that $M \leq N - \|\zeta\|_\infty$, since

$$\|u^\epsilon\|_{\text{sup}} \leq \|u_2\|_{\text{sup}} + \|u_1^\epsilon\|_{\text{sup}} \leq \|\zeta\|_\infty + \|u_1^\epsilon\|_\kappa,$$

we have,

$$\left\{\|u_1^\epsilon\|_\kappa < M\right\} \subset \left\{\|u^\epsilon\|_{\text{sup}} < N\right\} = \left\{\tau_N > T\right\}.$$

Recall the construction of v^ϵ and (3.14), and we can see that for any M satisfying $K_\kappa^0\epsilon\sqrt{N} \leq M \leq N - \|\zeta\|_\infty$,

$$P\left\{\|u_1^\epsilon\|_\kappa \geq M\right\} = P\left\{\|v^\epsilon\|_\kappa \geq M\right\} \leq \exp\left\{-K_\kappa^1\left(\frac{M}{\epsilon\sqrt{N}}\right)^2\right\}.$$

There exists N_0 such that for all $N > N_0$, $N - \|\zeta\|_\infty > K_\kappa^0\epsilon\sqrt{N}$. Then we can see that there is M_0 satisfying $M_0 \geq K_\kappa^0\epsilon\sqrt{M_0 + \|\zeta\|_\infty}$, and for any $M > M_0$, we have

$$P\left\{\|u_1^\epsilon\|_\kappa \geq M\right\} \leq \exp\left\{-\frac{K_\kappa^1}{\epsilon^2}\frac{M^2}{M + \|\zeta\|_\infty}\right\}. \tag{3.15}$$

Therefore, let $\{Q^\epsilon, \epsilon > 0\}$ be the family of probability measures on $C_\zeta^+\left([0,T]\times[0,L]\right)$ induced by $u_1^\epsilon(t,x)$, from the above estimate we know that $\{Q^\epsilon, \epsilon > 0\}$ is exponentially tight. Also, as pointed out before, $u^\epsilon(t,x) = u_2(t,x) + u_1^\epsilon(t,x)$, where $u_2(t,x)$ is a deterministic function. Then for any set $A \subset$

$C_\zeta^+([0,T]\times[0,L])$, define the set

$$A+u_2 := \{\phi : \phi = \psi + u_2, \psi \in A\}.$$

Then $A+u_2$ is a compact set in $C_\zeta^+([0,T]\times[0,L])$ if A is. Also since $Q^\epsilon(A) = P^\epsilon(A+u_2)$, we could conclude that the family of probability measures $\{P^\epsilon, \epsilon > 0\}$ induced on $C_\zeta^+([0,T]\times[0,L])$ by $u^\epsilon(t,x)$ is exponentially tight. □

Remark 3.2.5. We could see from this proof that we could generalize this exponential tightness result to the family of probability measures $\{P^\epsilon : \epsilon > 0\}$ on $C_\zeta^+([0,T]\times[0,L])$ induced by v^ϵ satisfies SPDE

$$\begin{aligned}
\partial_t v^\epsilon(t,x) &= \triangle v^\epsilon(t,x) + \epsilon\,(v^\epsilon(t,x))^p\,\ddot{W}_{tx} \\
v^\epsilon(0,x) &= \zeta(x)
\end{aligned}$$

where this SPDE is considered with either Dirichlet or Neumann boundary condition on domain $[0,T]\times[0,L]$, for any $0 < p < 1$. But for the case $p=1$ we will need a new estimate.

3.2.2 Proof of LDP

Given the LDP proved in [20], now we are going to prove a LDP result for the family of probability measures $\{P^\epsilon,\ \epsilon > 0\}$, which are probability measures induced on $C_\zeta^+([0,T]\times[0,L])$ by (3.1) and identify the rate function. Roughly speaking, we will get this LDP result through the following steps.

- Since we know that solution of (3.1) is continuous in both t and x almost surely, we could restrict the LDP for probability measures on $\mathcal{C}_\mu([0,T],\mathfrak{M})$ to Ω_0 which is properly defined below.
- Then we will establish a one-to-one map between this Ω_0 and $C_\zeta^+([0,T]\times[0,L])$. Induce the topology of Ω_0 to $C_\zeta^+([0,T]\times[0,L])$. Then we will have a LDP for probability measures on $C_\zeta^+([0,T]\times[0,L])$ but only when the space is equipped with a weaker topology not the sup-norm topology.
- Last we will use the exponential tightness result we proved in previous

section to prove the LDP for probability measures on $C^+_\zeta([0,T]\times[0,L])$ when the space is equipped with sup-norm topology.

Lemma 3.2.6. *Let Ω_0 be a set defined as*

$$\begin{aligned}\Omega_0 \quad := \quad & \{\nu \in \mathcal{C}_\mu([0,T],\mathfrak{M}) : \exists g(t,x) \in C^+_\zeta([0,T]\times[0,L]) \\ & s.t.\ \forall t \in [0,T]\ \nu_t(A) = \int_A g(t,x)dx, \forall A \in \mathscr{B}([0,L])\}.\end{aligned}$$

Define a map $k : C^+_\zeta([0,T]\times[0,L]) \to \Omega_0$ by

$$k(g(\cdot,x)) = \nu.\ if\ \forall t \in [0,T]\ \int_A g(t,x)dx = \nu_t(A)\ \forall A \in \mathscr{B}([0,L])$$

Then this map k is a bijection.

Proof. By the definition of Ω_0, k is surjective.

To prove k is injective, we would like to show that

$$g^1(\cdot,x) \neq g^2(\cdot,x) \Rightarrow \nu^1_\cdot \neq \nu^2_\cdot$$

where $g^1(t,x), g^2(t,x) \in C^+_\zeta([0,T]\times[0,L])$ and $k(g^1(\cdot,x)) = \nu^1_\cdot,\ k(g^2(\cdot,x)) = \nu^2_\cdot$. Also, by definitions $\nu^1_\cdot \neq \nu^2_\cdot$ if there is $t \in [0,T]$ such that for some bounded continuous function $h(x)$ on $[0,L]$, $\int_0^L h(x)d\nu^1_t(x) \neq \int_0^L h(x)d\nu^2_t(x)$.

Since both g^1 and g^2 are continuous with respect to x, if $\exists(t_0,x_0) \in [0,T]\times(0,L)$ such that $g^1(t_0,x_0) \neq g^2(t_0,x_0)$(without loss of generality, assume that $g^1(t_0,x_0) < g^2(t_0,x_0)$), then there exists an interval $B = (x_0 - r, x_0 + r)$ such that $\forall x \in B$, $g^1(t_0,x) < g^2(t_0,x)$. Taking $f(x) = \mathbf{1}_B(x)$, then we have

$$\int_0^L f(x)(g^2(t_0,x) - g^1(t_0,x))dx = \int_B (g^2(t_0,x) - g^1(t_0,x))dx > 0.$$

Meanwhile by standard procedure we know that $\exists\{h^n(x)\}_{n=1}^\infty$ which are bounded continuous functions on $[0,L]$ such that $\lim_{n\to\infty} h^n(x) = f(x)$ pointwisely. Then by Fatou's lemma,

$$\begin{aligned}& \int_0^L f(x)(g^2(t_0,x) - g^1(t_0,x))dx \\ \leq \quad & \lim_{n\to\infty} \int_0^L h^n(x)(g^2(t_0,x) - g^1(t_0,x))dx.\end{aligned}$$

Then we could conclude that $\exists h^n(x)$, a bounded continuous function on $[0, L]$,such that

$$\int_0^L h^n(x)(g^2(t_0, x) - g^1(t_0, x))dx > 0.$$

i.e. $\nu^1_{t_0} \neq \nu^2_{t_0}$ and hence $\nu^1_{\cdot} \neq \nu^2_{\cdot}$. When x_0 is on the boundary, we could use the same argument except to change the open interval B to some half open half closed interval. $\square$

Let τ^1 be the topology on $C^+_\zeta([0, T]\times[0, L])$ corresponding to convergence defined as follows, For any $\{g_n(t, x)\}_{n=1}^\infty \in C^+_\zeta([0, T] \times [0, L])$, $g_n \to g$ as $n \to \infty$ in τ^1 if

$$\sup_{(t,x)\in[0,T]\times[0,L]} |g_n(t, x) - g(t, x)| \to 0 \ \ as\ n \to \infty.$$

Since there is a one-to-one map between Ω_0 and $C^+_\zeta([0, T] \times [0, L])$, we could equip $C^+_\zeta([0, T] \times [0, L])$ with the same topology of Ω_0, which is the compact open topology induced by $\mathcal{C}_\mu([0, T], \mathfrak{M})$. We denote it by τ^2. This τ^2 topology corresponds to convergence defined as below, For any $\{g_n(t, x)\}_{n=1}^\infty \in C^+_\zeta([0, T]\times[0, L])$, $g_n \to g$ as $n \to \infty$ in τ^2 if for any bounded continuous function $f(x)$ on $[0, L]$,

$$\sup_{t\in[0,T]} |\int_0^L f(x)g_n(t, x)dx - \int_0^L f(x)g(t, x)dx| \to 0 \ \ as\ n \to \infty.$$

Regarding the two different topologies on the same space $C^+_\zeta([0, T] \times [0, L])$, we have the following two little lemmas to show their relation.

Lemma 3.2.7. *If $C \subset C^+_\zeta([0, T] \times [0, L])$ is closed in τ^2, then it is closed in τ^1. If $O \subset C^+_\zeta([0, T] \times [0, L])$ is open in τ^2, then it is open in τ^1.*

Proof. Let C be a closed set in τ^2, which means that if $g(t, x) \in C^+_\zeta([0, T] \times [0, L])$ is a τ_2 limit point of C, then $g \in C$. Define C_1 as the set of all τ_1 limit points of C and C_2 as the set of all τ_2 limit points of C. By hypothesis, we have $C_2 \subseteq C$. If we can show that $C_1 \subseteq C_2$, then $C_1 \subseteq C_2 \subseteq C$ and thus C is closed in τ^1.

To show $C_1 \subseteq C_2$, suppose that $g(t, x) \in C_1$, i.e. $\exists\{g_n(t, x)\}_{n=1}^\infty \in C$

such that

$$\sup_{(t,x)\in[0,T]\times[0,L]} |g_n(t,x) - g(t,x)| \to 0, \ as\ n \to \infty.$$

Then for any bounded continuous function $f(x)$ on $[0,L]$ we have

$$\begin{aligned} & \sup_{t\in[0,T]} |\int_0^L f(x)(g_n(t,x) - g(t,x))dx| \\ \leq & \int_0^t |f(x)| \sup_{(t,x)\in[0,T]\times[0,L]} |g_n(t,x) - g(t,x)|dx \to 0 \ as\ n \to \infty. \end{aligned}$$

i.e. $g_n(t,x) \to g(t,x)$ in τ_2, thus $g(t,x) \in C_2$ and we have shown $C_1 \subseteq C_2$.

The other half of the lemma is obviously true. $\square$

Lemma 3.2.8. *Suppose that $K \subseteq C^+_\zeta([0,T] \times [0,L])$ is compact under τ^1. If $C \subseteq C^+_\zeta([0,T] \times [0,L])$ is closed under τ^1, then $C \cap K$ is closed in τ^2. Also, if $O \subseteq C^+_\zeta([0,T] \times [0,L])$ is open under τ^1, then $O \cup K^c$ is open in τ^2.*

Proof. Take $\{g_n\}_{n=1}^\infty \in C \cap K$, such that $g_n \to g$ as $n \to \infty$ in τ_2. Our aim is to show that $g \in C \cap K$. Since $\{g_n\}_{n=1}^\infty \in K$ and K is compact under τ^1, we know that there exists a convergent subsequence $\{g_{n_j}\}$, such that $g_{n_j} \to \tilde{g}$ as $n_j \to \infty$ in τ_1. Then $\tilde{g} \in C \cap K$ since $C \cap K$ is a closed set in τ^1. Meanwhile, since convergence in τ_1 implies convergence in τ_2, $g_{n_j} \to \tilde{g}$ as $n_j \to \infty$ in τ_2. By the uniqueness of the limit of a sequence we could conclude that $g = \tilde{g}$ and thus $g \in C \cap K$. The second half of this lemma is obviously ture given the first part is proved. $\square$

Lemma 3.2.9. *Let Ω_0 be the set defined as in Lemma* 3.2.6. *Equip Ω_0 with the subspace topology induced by $\mathcal{C}_\mu([0,T],\mathfrak{M})$. Let $\{P^\epsilon,\ \epsilon > 0\}$ be the family of probability measures on $\mathcal{C}_\mu([0,T],\mathfrak{M})$ induced by X^ϵ which is defined in* (3.2)*, then $\{P^\epsilon,\ \epsilon > 0\}$ satisfies LDP on Ω_0 with a rate function $S(\nu)$,*

$$S(\nu) = \begin{cases} \frac{1}{2}\int_0^T \|\frac{d(\dot{\nu}_t - \triangle^*\nu_t)}{d\nu_t}\|^2_{L^2(\nu_t)} dt & if\ \nu \in H \\ \infty & otherwise \end{cases} \tag{3.16}$$

where this $S(\nu)$ is the rate function with which $\{P^\epsilon,\ \epsilon > 0\}$ satisfies LDP in $\mathcal{C}_\mu([0,T],\mathfrak{M})$. That is,

- $\liminf_{\epsilon\to 0} \epsilon^2 \ln P^\epsilon(O) \geq -\inf_{\nu\in O} S(\nu),\ for\ all\ open\ set\ O \subset \Omega_0;$

- $\limsup_{\epsilon\to 0}\epsilon^2 \ln P^\epsilon(C) \le -\inf_{\nu\in C} S(\nu),$ *for all closed set* $C \subset \Omega_0$.

Proof. Define the family of probability measures $\{Q^\epsilon : \epsilon > 0\}$ on $C_\zeta^+([0,T]\times[0,L])$ by $Q^\epsilon = P^\epsilon \circ k^{-1}$, where k is the one-to-one map defined in Lemma 3.2.6. Then $\{Q^\epsilon : \epsilon > 0\}$ is actually the family of probability measures induced by u^ϵ satisfying (3.1) on $C_\zeta^+([0,T]\times[0,L])$, which is exponentially tight by Lemma 3.2.4, i.e. for any $M > 0$, there exists a compact set $K_M \subset C_\zeta^+([0,T]\times[0,L])$, such that

$$\limsup_{\epsilon\to 0}\epsilon^2 \ln Q^\epsilon(K_M^c) \le -M.$$

For any $A \in C_\zeta^+([0,T]\times[0,L])$, let $k(A) := \left\{\nu \in \Omega_0 : k^{-1}(\nu) \in A\right\}$. Then

$$\limsup_{\epsilon\to 0}\epsilon^2 \ln P^\epsilon\left(k\left(K_M^c\right)\right) \le -M.$$

As K_M is a compact set in $C_\zeta^+([0,T]\times[0,L])$, then $k(K_M)$ would be a compact set in Ω_0, and since k is continuous, $k(K_M^c)$ is open in Ω_0. Then since $\{P^\epsilon : \epsilon > 0\}$ satisfies LDP in $\mathcal{C}_\mu([0,T],\mathfrak{M})$ [20], we could see that

$$\liminf_{\epsilon\to 0}\epsilon^2 \ln P^\epsilon\left(k\left(K_M^c\right)\right) \ge -\inf_{\nu\in k\left(K_M^c\right)} S(\nu).$$

Therefore we have shown that for any $M > 0$, there exists a compact set $K_M \subset C_\zeta^+([0,T]\times[0,L])$, such that

$$\inf_{\nu\in k\left(K_M^c\right)} S(\nu) \ge M.$$

As we have shown in Lemma 3.2.4, for any M, the set K_M would be a set of functions whose κ-Hölder norm is bounded shifted by a deterministic nonnegative continuous function. Therefore, we could see that $k(K_M) \subset \Omega_0$, for any M. So we have shown that

$$\inf_{\nu\in\Omega_0^c} S(\nu) = \infty.$$

From this we could see that for any set $\Gamma \subset \mathcal{C}_\mu([0,T],\mathfrak{M})$, $\inf_{\nu\in\Gamma} S(\nu) = \inf_{\nu\in(\Gamma\cap\Omega_0)} S(\nu)$. Since for any open set $O \subset \Omega_0$, $O = G\cap\Omega_0$ for some open set $G \in \mathcal{C}_\mu([0,T],\mathfrak{M})$, and for any closed set $C \subset \Omega_0$, $C = F\cap\Omega_0$ for some closed set $F \in \mathcal{C}_\mu([0,T],\mathfrak{M})$, the LDP holds for $\{P^\epsilon : \epsilon > 0\}$ on Ω_0 with rate function $S(\nu)$. $\square$

Remark 3.2.10. In Lemma 3.2.9, we proved that the family of probability measures $\{P^\epsilon,\ \epsilon > 0\}$ on Ω_0 induced by (3.2) satisfies LDP with a rate function $S(\nu)$. In Lemma 3.2.6, we proved that there is a one-to-one map $k^{-1} : \Omega_0 \to C_\zeta^+([0,T] \times [0,L])$ and later we introduced topology τ^2 on $C_\zeta^+([0,T] \times [0,L])$. This τ^2 topology makes k^{-1} continuous. By the contraction principle (see Theorem 4.2.1 in [1]) we could easily see that the family of probability measures $\{Q^\epsilon := P^\epsilon \circ k,\ \epsilon > 0\}$ on $C_\zeta^+([0,T] \times [0,L])$ satisfies LDP with a rate function $I(g) := \inf\{S(\nu) : \nu \in \Omega_0,\ g = k^{-1}(\nu)\} = S(k(g))$. And in fact, those Q^ϵ are probability measures induced by (3.1) on $C_\zeta^+([0,T] \times [0,L])$. Without confusion, we will use P^ϵ for them as well.

Before we state and prove our theorem about the LDP for $\{P^\epsilon,\ \epsilon > 0\}$ induced on $C_\zeta^+([0,T] \times [0,L])$ by (3.1) when the space is considered with topology τ^1, there is a last lemma we would like to prove, so that we can identify the rate function for certain functions in $C_\zeta^+([0,T] \times [0,L])$.

Lemma 3.2.11. *As described in Remark* 3.2.10*, for any $g \in C_\zeta^+([0,T] \times [0,L])$, define a function $I(g) = S(k(g))$, where for any $\nu \in \Omega_0$, $S(\nu)$ is defined in Lemma* 3.2.9 *and k is the one-to-one map defined in Lemma* 3.2.6*.Then*

$$I(g) = \begin{cases} \frac{1}{2}\int_0^T \int_0^L \frac{(\partial_t g - \triangle g)^2}{g} \mathbf{1}_{\{g>0\}} dxdt & \text{if } g \in W_2^{1,2},\ \text{satisfies same B.C. as } u^\epsilon \\ \infty & \text{otherwise} \end{cases} \tag{3.17}$$

where $W_2^{1,2}$ is the Sobolev space of functions on $[0,T] \times [0,L]$ with one square-integrable weak time derivative and two square-integrable weak space derivatives.

Proof. To prove this lemma, we need to show that $g \in W_2^{1,2}$ and it satisfies the same boundary condition as u^ϵ if and only if $k(g) \in H$, where H the space on which $S(\nu)$ is finite (We described this space in details in Section 3.1). We also need to show that when $g \in W_2^{1,2}$, satisfying the same boundary condition as u^ϵ, for $\nu = k(g)$, it is true that

$$\frac{1}{2}\int_0^T \| \frac{d(\dot{\nu}_t - \triangle^* \nu_t)}{d\nu_t} \|_{L^2(\nu_t)}^2 dt = \frac{1}{2}\int_0^T \int_0^L \frac{(\partial_t g - \triangle g)^2}{g} \mathbf{1}_{\{g>0\}} dxdt. \tag{3.18}$$

First we will show that if $g \in W_2^{1,2}$ and it satisfies the same boundary condition, then $\nu := k(g) \in H$ and (3.18) is true. To show that $\nu := k(g) \in H$, we need to show that the map $t \to \nu_t$ defined on $[0,T]$ is absolutely continuous, the generalized function $\dot{\nu}_t - \triangle^* \nu_t$ is absolutely continuous with respect to measure ν_t for almost all $t \in [0,T]$, and if we denote the Radon-Nikodym derivative by h_t, the map $t \to h_t$ belongs to $L^2(\nu) := L^2([0,T] \times [0,L], dr\nu_r(dy))$.

To see that $t \to \nu_t$ defined on $[0,T]$ is absolutely continuous, define an absolutely continuous function $f(t) = \int_0^L g(t,x)dx$ for all $t \in [0,T]$. Then for any test function $\varphi \in \Phi^{2+\gamma}$, (the space of $2+\gamma$ Hölder continuous functions $\varphi : [0,L] \to R$ that satisfy the same boundary condition as u^ϵ), $\|\varphi\|_{2+\gamma} \leq 1$, it is easy to see that

$$|\langle \nu_t, \varphi \rangle - \langle \nu_s, \varphi \rangle| = \left| \int_0^L \varphi(x) \left(g(t,x) - g(s,x) \right) dx \right| \leq |f(t) - f(s)| \,.$$

For any test function $\varphi \in \Phi^{2+\gamma}$, since φ and g satisfy the same boundary condition as u^ϵ, using integration by parts formula, we see that

$$\langle \triangle^* \nu_t, \varphi \rangle = \langle \nu_t, \triangle \varphi \rangle = \int_0^L \left(\triangle \varphi(x) \right) g(t,x) dx = \int_0^L \varphi(x) \left(\triangle g(t,x) \right) dx.$$

Therefore, for any test function $\varphi \in \Phi^{2+\gamma}$, it is true that

$$\langle \dot{\nu}_t - \triangle^* \nu_t, \varphi \rangle = \langle \partial_t g(t,x) - \triangle g(t,x), \varphi \rangle. \tag{3.19}$$

Now we define $h(t,x)$ as

$$h(t,x) = \begin{cases} \frac{\partial_t g(t,x) - \triangle g(t,x)}{g(t,x)} & \text{if } g(t,x) > 0 \\ 0 & \text{if } g(t,x) = 0 \end{cases} \tag{3.20}$$

Since $g(t,x) \in W_2^{1,2}$, $\partial_t g(t,x) = 0$ a.e. on the level set $\{(t,x) : g(t,x) = 0\}$ and so is $\triangle g(t,x)$ (see [22]). Then from (3.19), we see that $h(t,x)$ satisfies that for any test function $\varphi \in \Phi^{2+\gamma}$,

$$\langle \dot{\nu}_t - \triangle^* \nu_t, \varphi \rangle = \langle \nu_t, h(t,x) \varphi \rangle. \tag{3.21}$$

And at the same time

$$\int_0^T \int_0^L h^2(t,x)\nu_t(dx)dt = \int_0^T \int_0^L \frac{(\partial_t g - \triangle g)^2}{g} \mathbf{1}_{\{g(t,x)>0\}} dxdt < \infty. \tag{3.22}$$

i.e. we have shown that $\dot{\nu}_t - \triangle^* \nu_t$ is absolutely continuous with respect to measure ν_t for almost all $t \in [0,T]$, the Radon-Nikodym derivative h_t belongs to $L^2(\nu) := L^2([0,T] \times [0,L], dr\nu_r(dy))$, and (3.18) is true.

Next we will show that if $\nu := k(g) \in H$ then $g \in W_2^{1,2}$ and it satisfies the same boundary condition as u^ϵ. From the definition of domain H, we know that if $\nu := k(g) \in H$, then there exists a function $h(t,x)$ such that, for any test function $\varphi \in \Phi_{[0,L]}^{2+\gamma}$

$$\langle \dot{g}(t,x) - \triangle g(t,x), \varphi(t,x) \rangle = \langle g(t,x), h(t,x)\varphi(t,x) \rangle,$$

where $\dot{g}(t,x)$ and $\triangle g(t,x)$ are the distributional derivatives. This means that g solves equation

$$\partial_t g(t,x) - \triangle g(t,x) = h(t,x)g(t,x)$$

in distribution sense. Define function η such that it is smooth, compactly supported inside $(0,T) \times (0,L)$ and satisfies $\int_0^T \int_0^L \eta(t,x)dxdt = 1$. Then for any $\epsilon > 0$, define $\eta_\epsilon(t,x) = \frac{1}{\epsilon^2}\eta(t/\epsilon, x/\epsilon)$ and $g_\epsilon(t,x) = \int_0^T \int_0^L \eta_\epsilon(t-s, x-y)g(s,y)dyds$, we see that $g_\epsilon(t,x)$ is smooth and it satisfies the equation

$$\partial_t g_\epsilon(t,x) - \triangle g_\epsilon(t,x) = (h(t,x)g(t,x))_\epsilon$$

in the classical sense. Then using the a-priori estimate, we see that there exist constant M and M' such that

$$\begin{aligned} \|g_\epsilon\|^2_{W_2^{1,2}} &\leq M \int_0^T \int_0^L (\partial_t g_\epsilon(t,x) - \triangle g_\epsilon(t,x))^2 dxdt \\ &\leq M' \int_0^T \int_0^L h^2(t,x)g(t,x)dxdt. \end{aligned}$$

Since $h(t,x)$ belongs to $L^2(\nu) := L^2([0,T] \times [0,L], dr\nu_r(dy))$, the sequence of smooth functions $\{g_\epsilon(t,x)\}$ is bounded in $W_2^{1,2}$. Therefore there exists a subsequence converging to g and hence g is in $W_2^{1,2}$. To see that g satisfies the same boundary condition as u^ϵ, use the fact that $g \in W_2^{1,2}$, the definition of domain H and the integration by parts formula, we see that for any $\varphi \in \Phi^{2+\gamma}$, for a.e.$t \in [0,T]$

$$\int_0^L (\varphi \partial_t g - \varphi \triangle g)\, dx - g\varphi^{'}|_{x=0}^L + (\partial_x g)\varphi|_{x=0}^L = \int_0^L gh\varphi dx$$

where $\partial_t g$, $\triangle g$ and $\partial_x g$ are weak derivatives of g. Hence g must satisfies the same boundary condition as φ which satisfies the same boundary condition as u^ϵ to ensure the above equation holds for all test functions. $\square$

Lemma 3.2.12. *Let $\{P^\epsilon,\ \epsilon > 0\}$ be the family of probability measures induced on $C_\zeta^+([0,T]\times[0,L])$ by SPDE (3.1). It satisfies the large deviation principle with the good rate function I defined in the previous lemma.*

Proof. First, consider the upper bound. Suppose that $C \subseteq C_\zeta^+([0,T]\times[0,L])$ is a τ^1 closed set. Then for any τ^1 compact set K, $C\cap K$ is a τ^2 closed set. Then we have,

$$\begin{aligned}
\limsup_{\epsilon\to 0}\epsilon^2\ln P^\epsilon(C) &= \limsup_{\epsilon\to 0}\epsilon^2\ln(P^\epsilon(C\cap K)+P^\epsilon(C\cap K^c)) \\
&= \max[\limsup_{\epsilon\to 0}\epsilon^2\ln P^\epsilon(C\cap K), \limsup_{\epsilon\to 0}\epsilon^2\ln P^\epsilon(C\cap K^c)] \\
&\leq \max[-\inf_{\phi\in C\cap K} I(\phi), \limsup_{\epsilon\to 0}\epsilon^2\ln P^\epsilon(K^c)] \\
&\leq \max[-\inf_{\phi\in C} I(\phi), \limsup_{\epsilon\to 0}\epsilon^2\ln P^\epsilon(K^c)]
\end{aligned}$$

Since $\{P^\epsilon,\ \epsilon > 0\}$ is exponentially tight when $C_\zeta^+([0,T]\times[0,L])$ is equipped with τ^1, we could take K to be a compact set such that

$$\limsup_{\epsilon\to 0}\epsilon^2\ln P^\epsilon(K^c) < -M$$

for some $M > \inf_{\phi\in C} I(\phi)$. And therefore we have that

$$\limsup_{\epsilon\to 0}\epsilon^2\ln P^\epsilon(C) \leq -\inf_{\phi\in C} I(\phi)$$

Next consider the lower bound. Suppose that $O \subseteq C_\zeta^+([0,T]\times[0,L])$ is a τ^1 open set. Same as the first part, we have $O\cup K^c$ is open in τ^2 for any τ^1

compact set K. Then

$$\begin{aligned}\liminf_{\epsilon\to 0}\epsilon^2\ln P^\epsilon(O\cup K^c) &\le \liminf_{\epsilon\to 0}\epsilon^2\ln(P^\epsilon(O)+P^\epsilon(K^c))\\ &= \max[\liminf_{\epsilon\to 0}\epsilon^2\ln P^\epsilon(O),\liminf_{\epsilon\to 0}\epsilon^2\ln P^\epsilon(K^c)]\end{aligned}$$

Again, because of the exponential tightness of $\{P^\epsilon,\ \epsilon>0\}$ under τ^1, we take K to be the set such that

$$\liminf_{\epsilon\to 0}\epsilon^2\ln P^\epsilon(K^c)\le\limsup_{\epsilon\to 0}\epsilon^2\ln P^\epsilon(K^c)<-M'$$

for some $M'>\inf_{\phi\in O}I(\phi)$. By the lower bound in LDP for $\{P^\epsilon,\ \epsilon>0\}$ under τ^2, we have

$$\liminf_{\epsilon\to 0}\epsilon^2\ln P^\epsilon(O\cup K^c)\ge-\inf_{\phi\in O\cup K^c}I(\phi)\ge-\inf_{\phi\in O}I(\phi)$$

Thus,

$$\liminf_{\epsilon\to 0}\epsilon^2\ln P^\epsilon(O)\ge-\inf_{\phi\in O}I(\phi)$$

Finally, the goodness of the rate function can be concluded from the lower bound result and exponential tightness (see Lemma1.2.18 in [1]). $\square$

Chapter 4

Large Deviations for a Stochastic Reaction-Diffusion Equation

In this section, we consider the following one dimensional stochastic reaction-diffusion equation

$$
\begin{aligned}
\partial_t u^\epsilon(t,x) &= \partial_{xx} u^\epsilon(t,x) + f(u^\epsilon(t,x)) + \epsilon\sqrt{u^\epsilon(t,x)}\ddot{W}_{tx} \qquad (4.1)\\
u^\epsilon(0,x) &= \zeta(x), \;\; x \in [0,L]
\end{aligned}
$$

where the equation is considered on the domain $(t,x) \in [0,T] \times [0,L]$, W is a time-space white noise on a probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$, $\zeta(x) \geq 0$ is a continuous deterministic function on $[0,L]$ and f is a function satisfying certain conditions which will be specified later. This equation can be considered with periodic or Dirichlet or Neumann boundary conditions. As we discussed before, the boundary condition will have an effect on the definition of the solution.

It has been proved that if the initial condition ζ is continuous, and the function f satisfies the conditions listed below, then there exists a stochastic solution $u^\epsilon(t,x)$ living in the space $C^+_\zeta([0,T] \times [0,L])$, where $C^+_\zeta([0,T] \times [0,L])$ is the space of non-negative continuous functions on $[0,T] \times [0,L]$ such that for any $\phi(t,x) \in C^+_\zeta([0,T] \times [0,L])$, $\phi(0,x) = \zeta(x)$. Again, as $\epsilon \to 0$, we would expect the solution of (4.1) would tend to the solution of the corresponding deterministic reaction-diffusion equation. So, as in previous chapter, our aim is to find a rate function $I(\phi)$ and establish the LDP on the space $C_\zeta^+([0,T]\times[0,L])$ equipped with the sup-norm topology.

A similar problem is studied in [31]. In that paper, the author studied the stochastic partial differential equation

$$\partial_t u^\epsilon = \mathscr{L}u^\epsilon + a(x,u^\epsilon) + \epsilon\sigma(x,u^\epsilon)\ddot{W}_{tx}$$

where $\mathscr{L}$ is a time- and space- invariant second-order elliptic operator, a and σ are nice regular functions. But in his paper, he required the non-degeneracy of the noise term, i.e $\sigma \geq m > 0$ and that helped to simplify the proof of the LDP result.

This section will be divided into three parts. In the first part we will prove an exponential tightness result which will be used in many places later. Then in second and third part, we will prove the upper and lower bound in the LDP respectively.

In the proof of the upper bound, we will first use a Girsanov change of measure to transform (4.1) to a super-Brownian motion under the new probability measure. Then we will use the LDP for super-Brownian motion to derive an upper bound of the probability of any open ball in the space $C_\zeta^+([0,T]\times[0,L])$. Finally we will use this upper bound for the probability of open balls to prove the upper bound for general compact sets in the space $C_\zeta^+([0,T]\times[0,L])$ and the exponential tightness result will be sufficient for us to derive the upper bound for the probability of any general closed set in the space $C_\zeta^+([0,T]\times[0,L])$. In the proof of the lower bound, we will first use another Girsanov change of measure to prove an lower bound for the probability of open balls around strictly positive functions in the space $C_\zeta^+([0,T]\times[0,L])$. Then we prove a sort of continuity result for the rate function and use this to derive the lower bound for the probability of general open set in the space $C_\zeta^+([0,T]\times[0,L])$.

In the proof of the upper bound, our result depends on the LDP for super-Brownian motion, which we derived from results in [20]. There, they only proved the LDP result for super-Brownian motion with either Dirichlet or Neumann boundary conditions. Also, the technique we are going to use has some problems for Dirichlet boundary conditions. So although we believe that the upper bound should be true for all three different boundary conditions, we only managed to provide all details for Neumann boundary condition. In the case of lower bound, we managed to prove it independent of the result in [20]. But the method we are going to use won't cover the case for Dirichlet boundary condition either. Since in the first step we will prove an lower bound for the probability of open balls around strictly positive functions in $C_\zeta^+([0,T]\times[0,L])$, which contradicts with the Dirichlet boundary condition. So the lower bound estimate will be proved only for Neumann and periodic boundary conditions.

Although we can get an upper bound for the probabilities of open balls around strictly positive functions in $C_\zeta^+([0,T]\times[0,L])$, we still need to turn to the upper bound in the LDP for super Brownian motion. The reason is that we can't use open balls around strictly positive functions in $C_\zeta^+([0,T]\times[0,L])$ to cover general compact sets (which contains non-negative functions) and prove the sort of continuity result for the rate function (as we will do in the lower bound) at the same time.

Now we state here the conditions that function f needs to satisfy.

H1 There exists a constant F such that $|f(u)| \le F$, for any $u \in [0,\infty)$ and $f(0) = 0$,

H2 There exists a twice continuously differential function $\hat{f}(u)$ such that $f(u) = u\hat{f}(u)$,

H3 There exists a constant σ_1 such that $|u\hat{f}^2(u) - v\hat{f}^2(v)| \le \sigma_1|u - v|$, for any $u, v \in [0,\infty)$,

H4 $\hat{f}$ is bounded by a constant $\hat{F}$ and its first order derivative is bounded by a constant σ_2, its second order derivative is bounded by a constant $\hat{F}''$.

H5 f is Lipschitz with constant σ_3, i.e. $|f(u) - f(v)| \le \sigma_3|u - v|$ for any $u, v \in [0,\infty)$.

Note that the third and fifth condition above can be proved true given the others.

The result we will prove in this chapter is:

Theorem. *Suppose that $u^\epsilon(t,x)$ satisfies* (4.1) *with Neumann boundary conditions, and the function f satisfies conditions $H1 - H5$. Define the rate* *function* $I(\phi)$ *for* $\phi \in C_{\zeta}^{+}([0,T] \times [0,L])$ *as*

$$I(\phi) = \begin{cases} \frac{1}{2}\int_0^T \int_0^L \frac{(\partial_t \phi - \triangle\phi - f(\phi))^2}{\phi} \mathbf{1}_{\{\phi>0\}} dxdt & \text{if } \phi \in W_2^{1,2}, \text{ satisfies the same B.C. as } u^\epsilon \\ \infty & \text{otherwise} \end{cases} \tag{4.2}$$

where $W_2^{1,2}$ *is the Sobolev space of functions on* $[0,T]\times[0,L]$ *with one square-integrable time derivative and two square-integrable space derivatives. Let* P^ϵ *be the probability measure on* $C_{\zeta}^{+}([0,T] \times [0,L])$ *induced by* u^ϵ *and equip the space* $C_{\zeta}^{+}([0,T] \times [0,L])$ *with sup-norm topology. Then the family of probability measures* $\{P^\epsilon : \epsilon > 0\}$ *satisfies large deviation principles with the good rate function* I*. i.e.*

A1 *The level sets of* I*, which is*

$$\Phi(s) := \left\{\phi \in C_{\zeta}^{+}([0,T] \times [0,L]) : I(\phi) \le s\right\}$$

is compact in $C_{\zeta}^{+}([0,T] \times [0,L])$ *for each* $s \ge 0$*.*

A2 *For any closed set* $C \subset C_{\zeta}^{+}([0,T] \times [0,L])$*,*

$$\limsup_{\epsilon \to 0} \epsilon^2 \ln P^\epsilon(C) \le - \inf_{\phi \in C} I(\phi).$$

A3 *For any open set* $O \subset C_{\zeta}^{+}([0,T] \times [0,L])$*,*

$$\liminf_{\epsilon \to 0} \epsilon^2 \ln P^\epsilon(O) \ge - \inf_{\phi \in O} I(\phi).$$

4.1 An Exponential Tightness Result

We will use the same method as we used in proving the exponential tightness for Super Brownian motion.

Lemma 4.1.1. *Suppose that* u^ϵ *satisfies* (4.1) *with one of the three different boundary conditions: periodic, Dirichlet, or Nuemann. Suppose that* v^ϵ *satisfies*

$$\begin{aligned} \partial_t v^\epsilon(t,x) &= \partial_{xx} v^\epsilon(t,x) + f(v^\epsilon(t,x)) + \epsilon\sqrt{v^\epsilon(t,x)}\ddot{W}_{tx} \\ v^\epsilon(0,x) &= 0, \quad x \in [0,L] \end{aligned} \tag{4.3}$$

with the same boundary condition as u^ϵ*. Also suppose that the function* f

satisfies H1. For each $0 < \kappa < \frac{1}{4}$, there are positive constants K_κ^0, K_κ^1 depending only on κ, positive constant C_1 depending only on κ and F, a constant M_0 satisfying $M_0 - C_1 \geq K_\kappa^0 \epsilon \sqrt{M_0 - C_1 + \|\zeta\|_\infty}$, such that for any $M \geq M_0$

$$P\{\|v^\epsilon\|_\kappa \geq M\} \leq \exp\{-\frac{K_\kappa^1}{\epsilon^2}\frac{(M-C_1)^2}{M-C_1+\|\zeta\|_\infty}\}, \tag{4.4}$$

Also, the family of probability measures $\{P^\epsilon,\ \epsilon > 0\}$ induced by (4.1) on $C_\zeta^+([0,T]\times[0,L])$ is exponentially tight.

Proof. Let G be the Green's function for one dimensional heat equation considered on domain $[0,T]\times[0,L]$ with the same boundary condition as (4.1) is considered with. Then we could represent any solution of (4.1) as

$$\begin{aligned} u^\epsilon(t,x) &= \int_0^L G_t(x,y)\zeta(y)dy + \int_0^t\int_0^L G_{t-s}(x,y)f(u^\epsilon(s,y))dyds \\ &\quad + \int_0^t\int_0^L G_{t-s}(x,y)\epsilon\sqrt{u^\epsilon(s,y)}W(dyds) \\ &:= u_2(t,x) + u_3^\epsilon(t,x) + u_1^\epsilon(t,x). \end{aligned}$$

It is easy to see that

$$\begin{aligned} & |u_3^\epsilon(t,x) - u_3^\epsilon(s,y)| \\ \leq\ & \int_0^t\int_0^L |(G_{t-r}(x,z) - G_{s-r}(y,z))f(u^\epsilon(r,z))|dzdr \\ \leq\ & \{\int_0^t\int_0^L (G_{t-r}(x,z) - G_{s-r}(y,z))^2 dzdr\}^{1/2}\{\int_0^t\int_0^L f^2(u^\epsilon(r,z))dzdr\}^{1/2} \\ \leq\ & C_\kappa^1 F(TL)^{1/2}(r((t,x),(s,y)))^\kappa, \end{aligned}$$

where C_κ^1 is the constant appeared in Lemma 3.2.2. Therefore, there is a positive constant C_1 depending only on κ and F, such that $\|u_3^\epsilon\|_\kappa \leq C_1$.

It is easy to see that u_1^ϵ satisfies the equation

$$\begin{aligned} \partial_t u_1^\epsilon(t,x) &= \triangle u_1^\epsilon(t,x) + \epsilon\sqrt{u^\epsilon(t,x)}\ddot{W}_{tx} \\ u_1^\epsilon(0,x) &= 0,\ \ x\in[0,L] \end{aligned} \tag{4.5}$$

Then we could use the same argument as we used in Lemma 3.2.4 to see that there exists M_0 satisfying $M_0 - C_1 \geq K_\kappa^0 \epsilon \sqrt{M_0 - C_1 + \|\zeta\|_\infty}$, such that for any $M \geq M_0$,

$$P\{\|v^\epsilon\|_\kappa \geq M\} \leq P\{\|u_1^\epsilon\|_\kappa \geq M - C_1\} \leq \exp\{-\frac{K_\kappa^1}{\epsilon^2} \frac{(M - C_1)^2}{M - C_1 + \|\zeta\|_\infty}\}.$$

Then again use the same argument as we used in the proof of Lemma 3.2.4, we could conclude that the family of probability measures $\{P^\epsilon,\ \epsilon > 0\}$ induced by (4.1) on $C_\zeta^+([0,T] \times [0,L])$ is exponentially tight. $\square$

4.2 Lower Bound

In this section, we will prove the lower bound estimate in the LDP for the family of probability measures $\{P^\epsilon : \epsilon > 0\}$ on the space $C_\zeta^+([0,T] \times [0,L])$ induced by u^ϵ satisfying (4.1) with Neumann boundary conditions.

Lemma 4.2.1. *Suppose that $u(t,x)$ is a continuous function defined on $[0,T] \times [0,L]$ and $\varphi(t,x) \geq l > 0$ is a strictly positive smooth function defined on the same domain such that*

$$I(\varphi) = \frac{1}{2} \int_0^T \int_0^L \frac{(\partial_t \varphi - \triangle \varphi - f(\varphi))^2}{\varphi} \mathbf{1}_{\{\varphi > 0\}} dx dt = s,$$

for some $0 \leq s < \infty$. If $\|u - \varphi\|_{\sup} < \delta$, for some $\delta < l/2$, and f satisfies condition $H5$, then we have

$$\left| \int_0^T \int_0^L \frac{(\partial_t \varphi - \triangle \varphi - f(u))^2}{u} dx dt - 2I(\varphi) \right| \leq C_1(\varphi)\delta + C\delta^2, \tag{4.6}$$

where $C_1(\varphi) = \frac{4s}{l} + \frac{4\sigma_3 (2s\|\varphi\|_{L^1})^{1/2}}{l}$ and $C = \frac{2TL\sigma_3^2}{l}$.

Proof.

$$\begin{aligned}
&\left|\int_0^T\int_0^L \frac{(\partial_t\varphi-\triangle\varphi-f(u))^2}{u}dxdt-2I(\varphi)\right| \\
\leq\ & \int_0^T\int_0^L \left|\frac{(\partial_t\varphi-\triangle\varphi-f(\varphi)+f(\varphi)-f(u))^2}{u}-\frac{(\partial_t\varphi-\triangle\varphi-f(\varphi))^2}{\varphi}\right|dxdt \\
\leq\ & \int_0^T\int_0^L \left|\frac{(\partial_t\varphi-\triangle\varphi-f(\varphi))^2}{\varphi}\frac{\varphi-u}{u}\right|dxdt+\int_0^T\int_0^L\left|\frac{(f(\varphi)-f(u))^2}{u}\right|dxdt \\
&+\int_0^T\int_0^L\left|\frac{2(\partial_t\varphi-\triangle\varphi-f(\varphi))(f(\varphi)-f(u))}{u}\right|dxdt \\
\leq\ & \frac{4s}{l}\delta+\frac{2TL\sigma_3^2}{l}\delta^2+\frac{4\sigma_3(2s\|\varphi\|_{L^1})^{1/2}}{l}\delta \\
=\ & C_1(\varphi)\delta+C\delta^2.
\end{aligned}$$

In the last line above, we used Hölder inequality for the last term. We also used the fact that f is Lipschitz and $u(t,x)>l/2$ for any $(t,x)\in[0,T]\times[0,L]$ (since $\varphi(t,x)\geq l>0$ and $\|u-\varphi\|_{\sup}<\delta<l/2$). □

Lemma 4.2.2. *Suppose that $\varphi(t,x)\in C_\zeta^+([0,T]\times[0,L])$ is a strictly positive ($\varphi(t,x)\geq l>0$, for all $(t,x)\in[0,T]\times[0,L]$) smooth function satisfying Neumann boundary condition. Let u^ϵ be any solution to* (4.1) *where the function f satisfies condition $H1$ and $H5$. Then for any $l/2>\delta>0$, we have*

$$\liminf_{\epsilon\to 0}\epsilon^2\ln P(\{\|u^\epsilon-\varphi\|_{\sup}<\delta\})\geq -I(\varphi)-\frac{1}{2}C_1(\varphi)\delta-\frac{1}{2}C\delta^2,$$

where $C_1(\varphi)$ and C are defined as in previous lemma.

Proof. We shall use a Girsanov change of measure. Since we assume that φ is a smooth function on $[0,T]\times[0,L]$ and $\varphi(0,x)=\zeta(x)$, we could see that $Z^\epsilon(t,x):=u^\epsilon(t,x)-\varphi(t,x)$ satisfies the stochastic PDE

$$\begin{aligned}
\partial_t Z^\epsilon(t,x) &= \triangle Z^\epsilon(t,x)-\{\partial_t\varphi(t,x)-\triangle\varphi(t,x)-f(u^\epsilon(t,x))\} \qquad (4.7)\\
&\quad+\epsilon\sqrt{u^\epsilon(t,x)}\ddot{W}_{tx}, \\
Z^\epsilon(0,x) &= 0.
\end{aligned}$$

Let

$$N^\epsilon(t,x) = \frac{\partial_t \varphi(t,x) - \triangle\varphi(t,x) - f(u^\epsilon(t,x))}{\epsilon\sqrt{u^\epsilon(t,x)}}\mathbf{1}_{\left\{u^\epsilon(t,x) > \frac{1}{2}l\right\}}.$$

Define a new measure Q on $(\Omega, \mathscr{F})$ by

$$\begin{aligned} \frac{dQ}{dP} &= \exp\left\{\int_0^T \int_0^L N^\epsilon(t,x) W(dxdt) - \frac{1}{2}\int_0^T \int_0^L \left(N^\epsilon(t,x)\right)^2 dxdt\right\} \\ &:= \mathcal{E}_T. \end{aligned}$$

We could see that the process $\mathcal{E}_t$ is a non-negative local martingale and hence a supermartingale. Therefore to check that it is a true martingale we only need to check that $E^P\left[\mathcal{E}_t\right] \geq 1$. We could define a stopping time

$$\tau_N := \inf\left\{t : \int_0^t \int_0^L \left(N^\epsilon(t,x)\right)^2 dxdt \geq N\right\}.$$

Then for each N, the stopped process $\mathcal{E}_{t\wedge\tau_N}$ is a true martingale. Furthermore we could see that

$$E^P\left[\mathcal{E}_t\right] \geq E^P\left[\mathcal{E}_{t\wedge\tau_N}\mathbf{1}_{\{\tau_N \geq t\}}\right] = E^P\left[\mathcal{E}_{t\wedge\tau_N}\right] - E^P\left[\mathcal{E}_{t\wedge\tau_N}\mathbf{1}_{\{\tau_N < t\}}\right].$$

Since $\lim_{N\to\infty} E^P\left[\mathcal{E}_{t\wedge\tau_N}\mathbf{1}_{\{\tau_N<t\}}\right] = \lim_{N\to\infty} Q\left(\tau_N < t\right) = 0$, we could see that $E^P\left[\mathcal{E}_t\right] \geq 1$ and hence $\mathcal{E}_t$ is a true martingale. Therefore, Q is a properly defined Girsanov change of measure.

Then define a new noise process $\widetilde{W}$ based on $[0,T]\times[0,L]$ as

$$\widetilde{W}(B) = W(B) - \int_0^T \int_0^L \mathbf{1}_{(B)} N^\epsilon(t,x) dxdt, \tag{4.8}$$

for any $B \in \mathscr{B}\left([0,T]\times[0,L]\right)$. We could use Girsanov theorem to see that $\widetilde{W}$ is a Q-Brownian sheet. Let

$$A = \left\{\|u^\epsilon - \varphi\|_{\sup} < \delta\right\}.$$

For $\delta < l/2$, we have

$$
\begin{aligned}
& P(A) \qquad (4.9) \\
= \ & E^Q\left\{\mathbf{1}_{\{A\}} \exp\left[-\int_0^T \int_0^L N^\epsilon(t,x) W(dxdt) + \frac{1}{2}\int_0^T \int_0^L \left(N^\epsilon(t,x)\right)^2 dxdt\right]\right\} \\
= \ & E^Q\left\{\mathbf{1}_{\{A\}} \exp\left[-I_T^\epsilon - \frac{1}{2}\langle I^\epsilon\rangle_T\right]\right\},
\end{aligned}
$$

where

$$
I_T^\epsilon = \frac{1}{\epsilon}\int_0^T \int_0^L \frac{\partial_t\varphi(t,x) - \triangle\varphi(t,x) - f(u^\epsilon(t,x))}{\sqrt{u^\epsilon(t,x)}} \mathbf{1}_{\{u^\epsilon(t,x) > l/2\}} \widetilde{W}(dxdt).
$$

Using Lemma 4.2.1, we could see that on set A,

$$
\begin{aligned}
-\frac{1}{2}\langle I^\epsilon\rangle_T &= -\frac{1}{2\epsilon^2}\int_0^T \int_0^L \frac{\left(\partial_t\varphi - \triangle\varphi - f(u^\epsilon)\right)^2}{u^\epsilon} dxdt \qquad (4.10) \\
&\geq -\frac{1}{\epsilon^2}\left[I(\varphi) + \frac{1}{2}C_1(\varphi)\delta + \frac{1}{2}C\delta^2\right].
\end{aligned}
$$

Using Jensen's inequality, we have that

$$
E^Q\left\{\mathbf{1}_{\{A\}} \exp\left[-I_T^\epsilon\right]\right\} \geq Q\left(A\right) \exp\left\{\frac{E^Q\left[\mathbf{1}_{\{A\}}\left(-I_T^\epsilon\right)\right]}{Q\left(A\right)}\right\}.
$$

And using Cauchy-Schwarz inequality, we have that

$$
\begin{aligned}
\exp\left\{\frac{E^Q\left[\mathbf{1}_{\{A\}}\left(-I_T^\epsilon\right)\right]}{Q\left(A\right)}\right\} &\geq \exp\left\{-\left[\frac{E^Q\left[\mathbf{1}_{\{A\}}\left(I_T^\epsilon\right)^2\right]}{Q\left(A\right)}\right]^{1/2}\right\} \qquad (4.11) \\
&\geq \exp\left\{-\left[\frac{E^Q\left[\left(I_T^\epsilon\right)^2\right]}{Q\left(A\right)}\right]^{1/2}\right\}.
\end{aligned}
$$

Since $\widetilde{W}$ is a Q-Brownian sheet, we have

$$
E^Q\left[\left(I_T^\epsilon\right)^2\right] = \frac{1}{\epsilon^2}\int_0^T \int_0^L E^Q\left[\frac{\left(\partial_t\varphi - \triangle\varphi - f(u^\epsilon)\right)^2}{u^\epsilon}\mathbf{1}_{\{u^\epsilon > l/2\}}\right] dxdt.
$$

Since f is bounded, φ is smooth, there exists $C_2(\varphi) < \infty$, such that $E^Q\left[(I_T^\epsilon)^2\right] \le \frac{1}{\epsilon^2}C_2(\varphi)$. Collecting all the above inequalities, we see that

$$P(A) \ge \exp\left[-\frac{1}{\epsilon^2}\left(I(\varphi)+\frac{1}{2}C_1(\varphi)\delta+\frac{1}{2}C\delta^2\right)\right]Q(A)\exp\left[-\frac{1}{\epsilon}\left(\frac{C_2(\varphi)}{Q(A)}\right)^{1/2}\right]. \tag{4.12}$$

If we can prove that

$$\lim_{\epsilon\longrightarrow 0} Q\left(A\right) = 1, \tag{4.13}$$

then the result is true, i.e. we have

$$\liminf_{\epsilon\longrightarrow 0} \epsilon^2 \ln P\left(\{\|u^\epsilon - \varphi\|_{\sup} < \delta\}\right) \ge -I(\varphi) - \frac{1}{2}C_1(\varphi)\delta - \frac{1}{2}C\delta^2.$$

The rest of this proof will be used to prove (4.13). Again by Girsanov theorem, we could see that under Q, Z^ϵ satisfies the following stochastic partial differential equation with Neumann boundary condition,

$$\begin{aligned}
\partial_t Z^\epsilon(t,x) &= \triangle Z^\epsilon(t,x) + \epsilon\sqrt{u^\epsilon(t,x)}\ddot{\widetilde{W}}_{tx} \\
&\quad + \left[\partial_t\varphi(t,x) - \triangle\varphi(t,x) - f(u^\epsilon(t,x))\right]\left(\mathbf{1}_{\{u^\epsilon(t,x)>l/2\}} - 1\right), \\
Z^\epsilon(0,x) &= 0.
\end{aligned} \tag{4.14}$$

Suppose that $\hat{Z}^\epsilon$ satisfies the following SPDE with Neumann boundary condition

$$\begin{aligned}
\partial_t \hat{Z}^\epsilon(t,x) &= \triangle \hat{Z}^\epsilon(t,x) + \epsilon\sqrt{u^\epsilon(t,x)}\ddot{\widetilde{W}}_{t,x}, \\
\hat{Z}^\epsilon(0,x) &= 0.
\end{aligned} \tag{4.15}$$

Define a stopping time $\tau := \inf\left\{t \ge 0 : \inf_{x\in[0,L]} u^\epsilon(t,x) < l/2\right\}$. Then let $Y^\epsilon(t,x)$ be defined as

$$Y^\epsilon(t,x) = \begin{cases} Z^\epsilon(t,x) & \text{if } t \le \tau, \\ \int_0^L G_{t-\tau}(x,y)Z^\epsilon(\tau,y)dy + \epsilon\int_{t-\tau}^t\int_0^L G_{t-s}(x,y)\sqrt{u^\epsilon(s,x)}\ddot{\widehat{W}}(dxdt) & \text{if } t > \tau, \end{cases}$$

where G is the Green's function for one dimensional Laplacian, and $\widehat{W}$ is a Q-Brownian sheet independent of $\widetilde{W}$. By the uniqueness in law of the solution to (4.15), we could say that Y^ϵ and $\hat{Z}^\epsilon$ have the same law. Recall that

$$A = \{\|u^\epsilon - \varphi\|_{\sup} < \delta\} = \{\|Z^\epsilon\|_{\sup} < \delta\}.$$

Since $\varphi(t,x) \geq l$ for all $(t,x) \in [0,T] \times [0,L]$ and $\delta < l/2$, it is easy to see that if $\tau \leq T$, then for some $x \in [0,L]$, $|\varphi(\tau,x) - u^\epsilon(\tau,x)| \geq l/2$. i.e. we have that on the set A, $\tau > T$. Therefore, we have that

$$Q(A) = Q\left(\|Z^\epsilon\|_{\sup} < \delta,\ \tau > T\right) = Q\left(\|Y^\epsilon\|_{\sup} < \delta\right) = Q\left(\left\|\hat{Z}^\epsilon\right\|_{\sup} < \delta\right). \tag{4.16}$$

Let $\bar{Z}^\epsilon(t,x)$ satisfies the following equation with Neumann boundary condition

$$\begin{aligned} \partial_t \bar{Z}^\epsilon(t,x) &= \triangle \bar{Z}^\epsilon(t,x) + \epsilon\left(\sqrt{u^\epsilon(t,x)} \wedge \frac{1}{\sqrt{\epsilon}}\right)\ddot{\widetilde{W}}_{tx}, \\ \bar{Z}^\epsilon(0,x) &= 0. \end{aligned}$$

Then by the conclusion of Remark 3.2.3 and Lemma 3.2.1, we could see that for any δ, there exists $K^0_\kappa > 0$, $K^1_\kappa > 0$, ϵ^* such that for all $\epsilon \leq \epsilon^*$, $\delta \geq K^0_\kappa\sqrt{\epsilon}$, and such that $Q\left(\left\|\bar{Z}^\epsilon\right\|_{\sup} \geq \delta\right) \leq \exp\left(-K^1_\kappa\left(\frac{\delta^2}{\epsilon}\right)\right)$. i.e. we have

$$\lim_{\epsilon \longrightarrow 0} Q\left(\left\|\bar{Z}^\epsilon\right\|_{\sup} \geq \delta\right) = 0.$$

Meanwhile we could see that

$$\begin{aligned} Q\left(\left\|\hat{Z}^\epsilon\right\|_{\sup} \geq \delta\right) &= Q\left(\left\|\hat{Z}^\epsilon\right\|_{\sup} \geq \delta,\ \left\|\bar{Z}^\epsilon\right\|_{\sup} \geq \delta,\ \|u^\epsilon\|_{\sup} \geq \frac{1}{\epsilon}\right) \\ &\quad + Q\left(\left\|\hat{Z}^\epsilon\right\|_{\sup} \geq \delta,\ \left\|\bar{Z}^\epsilon\right\|_{\sup} \geq \delta,\ \|u^\epsilon\|_{\sup} < \frac{1}{\epsilon}\right) \\ &\quad + Q\left(\left\|\hat{Z}^\epsilon\right\|_{\sup} \geq \delta,\ \left\|\bar{Z}^\epsilon\right\|_{\sup} < \delta,\ \|u^\epsilon\|_{\sup} \geq \frac{1}{\epsilon}\right) \\ &\quad + Q\left(\left\|\hat{Z}^\epsilon\right\|_{\sup} \geq \delta,\ \left\|\bar{Z}^\epsilon\right\|_{\sup} < \delta,\ \|u^\epsilon\|_{\sup} < \frac{1}{\epsilon}\right) \\ &\leq 2Q\left(\left\|\bar{Z}^\epsilon\right\|_{\sup} \geq \delta\right) + Q\left(\|u^\epsilon\|_{\sup} \geq \frac{1}{\epsilon}\right). \end{aligned}$$

In order to estimate $Q\left(\|u^\epsilon\|_{\sup} \geq \frac{1}{\epsilon}\right)$, we see that under Q, u^ϵ satisfies the following equation with Neumann boundary condition,

$$\begin{aligned}
\partial_t u^\epsilon(t,x) &= \triangle u^\epsilon(t,x)+\epsilon\sqrt{u^\epsilon(t,x)}\widetilde{\ddot{W}}_{tx} \\
&\quad +f(u^\epsilon(t,x))+\left[\partial_t\varphi(t,x)-\triangle\varphi(t,x)-f(u^\epsilon(t,x))\right]\mathbf{1}_{\{u^\epsilon(t,x)>l/2\}}, \\
u^\epsilon(0,x) &= \zeta(x).
\end{aligned}$$

Suppose u_1^ϵ satisfies the same SPDE but with initial condition $u_1^\epsilon(0,x)\equiv 0$, then $u^\epsilon(t,x)=u_1^\epsilon(t,x)+\int_0^L G_t(x,y)\zeta(y)dy$. Since the drift term in the SPDE above is bounded for any fixed smooth function φ, and the initial condition $\zeta(x)$ is a continuous function on $[0,L]$, there would be an estimate on the Hölder norm of u_1^ϵ similar to (4.4). Then, for ϵ small enough, since

$$Q\left(\|u^\epsilon\|_{\sup}\geq\frac{1}{\epsilon}\right)\leq Q\left(\|u_1^\epsilon\|_{\sup}\geq\frac{1}{\epsilon}-\|\zeta\|_\infty\right)\leq Q\left(\|u_1^\epsilon\|_\kappa\geq\frac{1}{\epsilon}-\|\zeta\|_\infty\right),$$

we could conclude that

$$\lim_{\epsilon\to 0}Q\left(\|u^\epsilon\|_{\sup}\geq\frac{1}{\epsilon}\right)=0.$$

Therefore, we know that

$$\lim_{\epsilon\to 0}Q\left(\left\|\hat{Z}^\epsilon\right\|_{\sup}<\delta\right)=1,$$

and recall (4.16), we have proved that

$$\lim_{\epsilon\to 0}Q\left(A\right)=1.$$

□

Now we would like to prove the lower bound for general open set by using the lower bound for open balls, which we just proved. Let O be an open set in $C_\zeta^+\left([0,T]\times[0,L]\right)$ when the space is equipped with the sup-norm topology. Lower bound in LDP, i.e.

$$\liminf_{\epsilon\to 0}\epsilon^2\ln P^\epsilon(O)\geq-\inf_{\varphi\in O}I(\varphi)$$

will be true if we can prove for any $\varphi\in O$, such that $I(\varphi)=s<\infty$, it is true that

$$\liminf_{\epsilon\to 0} \epsilon^2 \ln P^\epsilon(O) \geq -I(\varphi). \tag{4.17}$$

Since O is open, for any $\varphi \in O$, there exists $\delta_1 > 0$ such that for all $0 < \delta < \delta_1$, open balls $B(\varphi, \delta) \subset O$. Then

$$\liminf_{\epsilon\to 0} \epsilon^2 \ln P^\epsilon(O) \geq \liminf_{\epsilon\to 0} \epsilon^2 \ln P\left(\|u^\epsilon(t,x) - \varphi(t,x)\|_{\sup} < \delta\right).$$

If we could apply Lemma 4.2.2 here, we should see that

$$\liminf_{\epsilon\to 0} \epsilon^2 \ln P^\epsilon(O) \geq -I(\varphi) - \frac{1}{2}C_1(\varphi)\delta - \frac{1}{2}C\delta^2,$$

for all $0 < \delta < \delta_1$. Then we would let δ go to 0 and get 4.17. But the problem is that we only proved Lemma 4.2.2 for φ strictly positive and smooth. So the above argument doesn't hold for $\varphi \in O$ which is not strictly positive or not smooth. Roughly speaking, in order to use Lemma 4.2.2 to prove (4.17) for those φ which are not strictly positive or not smooth, we need to lift and smooth φ to get an "improved" function φ_δ, and make sure $I(\varphi_\delta)$ stays close to $I(\varphi)$. The following lemma will show that this can be achieved.

Lemma 4.2.3. *Suppose O is an open set in the space $C_\zeta^+([0,T]\times[0,L])$. Let $\varphi \in O$ satisfying $I(\varphi) = s < \infty$. Then the following three statements are true.*

1. *There exists $\delta_1 < 1$ such that for all $0 < \delta < \delta_1$, open balls $B(\varphi, \delta) \subset O$,*

2. *For each of those δ, there exists a strictly positive and smooth function φ_δ and $\delta_1^{'} = \frac{2}{3}\delta - \frac{1}{3}\delta^2$, such that for any $0 < \delta^{'} < \delta_1^{'}$, $B(\varphi_\delta, \delta^{'}) \subset B(\varphi, \delta)$,*

3. *For those φ_δ, we have*

$$\lim_{\delta\to 0} I(\varphi_\delta) = I(\varphi).$$

Proof. The first statement is obviously true by the definition of open sets.

To prove the second statement, we construct the function φ_δ as follows. Since $I(\varphi) = s < \infty$, by the definition of $I(\varphi)$, we know that $\varphi \in W_2^{1,2}$ and $\varphi \geq 0$. So for any $0 < \delta < \delta^1$, we define $\tilde{\varphi}_\delta = \varphi + \delta/3$. Then $\tilde{\varphi}_\delta \in W_2^{1,2}$ and $\tilde{\varphi}_\delta > 0$. Next, since smooth functions are dense in $W_2^{1,2}$, there exists φ_δ

such that $\|\tilde{\varphi}_\delta - \varphi_\delta\|_{W_2^{1,2}} \leq \delta^2/3$. Also, since $\varphi_\delta \geq \delta/3 - \delta^2/3$ and $0 < \delta < 1$, this φ_δ is strictly positive as well. Finally, let $\delta_1^{'} = \frac{2}{3}\delta - \frac{1}{3}\delta^2$, then for any $0 < \delta^{'} < \delta_1^{'}$, if $\phi \in B(\varphi_\delta, \delta^{'})$, then

$$|\phi - \varphi| \leq |\phi - \varphi_\delta| + |\varphi_\delta - \tilde{\varphi}_\delta| + |\tilde{\varphi}_\delta - \varphi| \leq \delta.$$

So, we have $B(\varphi_\delta, \delta^{'}) \subset B(\varphi, \delta)$.

We will use the same idea to prove the last statement. We will use the fact that $|I(\varphi) - I(\varphi_\delta)| \leq |I(\varphi) - I(\tilde{\varphi}_\delta)| + |I(\tilde{\varphi}_\delta) - I(\varphi_\delta)|$ and show that

$$\begin{aligned} \lim_{\delta \to 0} |I(\varphi) - I(\tilde{\varphi}_\delta)| &= 0, \\ \lim_{\delta \to 0} |I(\tilde{\varphi}_\delta) - I(\varphi_\delta)| &= 0. \end{aligned}$$

i.e. neither lifting (getting $\tilde{\varphi}_\delta$ from φ) nor smoothing (getting φ_δ from $\tilde{\varphi}_\delta$) will change the value of I too much.

We can see that

$$\begin{aligned} & 2\,|I(\tilde{\varphi}_\delta) - I(\varphi)| \\ \leq & \int_0^T \int_0^L \left| \frac{(\partial_t \tilde{\varphi}_\delta - \triangle \tilde{\varphi}_\delta - f(\tilde{\varphi}_\delta))^2 \mathbf{1}_{\{\tilde{\varphi}_\delta > 0\}}}{\tilde{\varphi}_\delta} - \frac{(\partial_t \varphi - \triangle \varphi - f(\varphi))^2 \mathbf{1}_{\{\varphi > 0\}}}{\varphi} \right| dxdt \\ \leq & \int_0^T \int_0^L \left| \frac{(\partial_t \varphi - \triangle \varphi - f(\varphi + \delta/3))^2}{\varphi + \delta/3} - \frac{(\partial_t \varphi - \triangle \varphi - f(\varphi))^2 \mathbf{1}_{\{\varphi > 0\}}}{\varphi + \delta/3} \right| dxdt \\ & + \int_0^T \int_0^L \left| \frac{(\partial_t \varphi - \triangle \varphi - f(\varphi))^2 \mathbf{1}_{\{\varphi > 0\}}}{\varphi + \delta/3} - \frac{(\partial_t \varphi - \triangle \varphi - f(\varphi))^2 \mathbf{1}_{\{\varphi > 0\}}}{\varphi} \right| dxdt \\ := & E_1 + E_2. \end{aligned}$$

Since $\varphi \geq 0$, we have,

$$E_2 = \int_0^T \int_0^L \left| \frac{(\partial_t \varphi - \triangle \varphi - f(\varphi))^2}{\varphi} \frac{\frac{\delta}{3} \mathbf{1}_{\{\varphi > 0\}}}{\varphi + \delta/3} \right| dxdt \leq 2I(\varphi) = 2s.$$

Then by monotone convergence theorem we could see that $\lim_{\delta \to 0} E_2 = 0$.

Meanwhile, since f is differentiable on $[0, \infty)$ and its derivative is bounded by $F^{'}$, we can see that there exists $\xi(t, x) \in (\varphi(t, x), \varphi(t, x) + \delta/3)$, such that

$f\left(\varphi(t,x)+\delta/3\right)=f\left(\varphi(t,x)\right)+f^{'}\left(\xi(t,x)\right)\frac{\delta}{3}$. Then we have

$$
\begin{aligned}
E_1 &= \int_0^T\int_0^L \left|\frac{\left(\partial_t\varphi-\triangle\varphi-f(\varphi)-f^{'}(\xi)\delta/3\right)^2-\left(\partial_t\varphi-\triangle\varphi-f(\varphi)\right)^2\mathbf{1}_{\{\varphi>0\}}}{\varphi+\delta/3}\right| dxdt \\
&\leq \int_0^T\int_0^L \left|\frac{\left(\partial_t\varphi-\triangle\varphi-f\left(\varphi\right)\right)^2\left(1-\mathbf{1}_{\{\varphi>0\}}\right)}{\varphi+\delta/3}\right| dxdt \\
&\quad +\int_0^T\int_0^L \left|\frac{-2f^{'}(\xi)(\delta/3)(\partial_t\varphi-\triangle\varphi-f(\varphi))+(f^{'}(\xi)\delta/3)^2}{\varphi+\delta/3}\right| dxdt
\end{aligned}
$$

Since $\varphi\in W_2^{1,2}$, on the level set $\{(t,x):\varphi(t,x)=0\}$, we have $\partial_t\varphi(t,x)=\triangle\varphi(t,x)=0$. Further since $f(0)=0$, we know that

$$
\int_0^T\int_0^L \left|\frac{\left(\partial_t\varphi-\triangle\varphi-f\left(\varphi\right)\right)^2\left(1-\mathbf{1}_{\{\varphi>0\}}\right)}{\varphi+\delta/3}\right| dxdt=0.
$$

Therefore

$$
\begin{aligned}
E_1 &\leq \int_0^T\int_0^L \left|\frac{-2f^{'}(\xi)(\delta/3)(\partial_t\varphi-\triangle\varphi-f(\varphi))+(f^{'}(\xi)\delta/3)^2}{\varphi+\delta/3}\right| dxdt \\
&\leq 2F^{'}\frac{\delta}{3}\int_0^T\int_0^L\left|\frac{\partial_t\varphi-\triangle\varphi-f(\varphi)}{\varphi+\delta/3}\right| dxdt+\int_0^T\int_0^L\frac{(F^{'})^2\delta^2/9}{\varphi+\delta/3}dxdt \\
&\leq \frac{2}{3}F^{'}\sqrt{TL}\delta\sqrt{\int_0^T\int_0^L\frac{(\partial_t\varphi-\triangle\varphi-f(\varphi))^2}{(\varphi+\delta/3)^2}dxdt}+\frac{1}{3}TL\left(F^{'}\right)^2\delta \\
&\leq \frac{2}{3}F^{'}\sqrt{TL}\delta\sqrt{\int_0^T\int_0^L\frac{(\partial_t\varphi-\triangle\varphi-f(\varphi))^2}{\varphi(\delta/3)}dxdt}+\frac{1}{3}TL\left(F^{'}\right)^2\delta \\
&\leq \left(\frac{8}{3}TLs\right)^{1/2}F^{'}\delta^{1/2}+\frac{1}{3}TL\left(F^{'}\right)^2\delta.
\end{aligned}
$$

So we have $\lim_{\delta\to 0}E_1=0$ and hence

$$
\lim_{\delta\to 0}\left|I(\varphi)-I(\tilde{\varphi}_\delta)\right|=0.
$$

Next, similarly, we see that

$$\begin{aligned}
& 2\left|I(\tilde{\varphi}_\delta)-I(\varphi_\delta)\right| \\
\leq\ & \int_0^T\int_0^L \left|\frac{\left(\partial_t\tilde{\varphi}_\delta-\triangle\tilde{\varphi}_\delta-f(\tilde{\varphi}_\delta)\right)^2\mathbf{1}_{\{\tilde{\varphi}_\delta>0\}}}{\tilde{\varphi}_\delta}-\frac{\left(\partial_t\varphi_\delta-\triangle\varphi_\delta-f(\varphi_\delta)\right)^2\mathbf{1}_{\{\varphi_\delta>0\}}}{\varphi_\delta}\right|dxdt \\
\leq\ & \int_0^T\int_0^L \left|\frac{\left(\partial_t\tilde{\varphi}_\delta-\triangle\tilde{\varphi}_\delta-f(\tilde{\varphi}_\delta)\right)^2}{\tilde{\varphi}_\delta}-\frac{\left(\partial_t\tilde{\varphi}_\delta-\triangle\tilde{\varphi}_\delta-f(\tilde{\varphi}_\delta)\right)^2}{\varphi_\delta}\right|dxdt \\
& +\int_0^T\int_0^L \left|\frac{\left(\partial_t\tilde{\varphi}_\delta-\triangle\tilde{\varphi}_\delta-f(\tilde{\varphi}_\delta)\right)^2}{\varphi_\delta}-\frac{\left(\partial_t\varphi_\delta-\triangle\varphi_\delta-f(\varphi_\delta)\right)^2}{\varphi_\delta}\right|dxdt \\
:=\ & E_3+E_4.
\end{aligned}$$

Since $\sup_{(t,x)\in[0,T]\times[0,L]}|\varphi(t,x)-\tilde{\varphi}_\delta(t,x)|\leq C(T,L)\left\|\varphi-\tilde{\varphi}_\delta\right\|_{W_2^{1,2}}\leq C(T,L)\delta^2/3$, where $C(T,L)$ is a constant depends only on T and L, and $\varphi_\delta\geq\delta/3-\delta^2/3$, we have

$$\begin{aligned}
E_3 &\leq \int_0^T\int_0^L \frac{\left(\partial_t\tilde{\varphi}_\delta-\triangle\tilde{\varphi}_\delta-f(\tilde{\varphi}_\delta)\right)^2}{\tilde{\varphi}_\delta}\left|\frac{\varphi_\delta-\tilde{\varphi}_\delta}{\varphi_\delta}\right|dxdt \\
&\leq \frac{C(T,L)\delta^2/3}{\delta/3-\delta^2/3}2I(\tilde{\varphi}_\delta).
\end{aligned}$$

Therefore, $\lim_{\delta\to0}E_3\leq\lim_{\delta\to0}\frac{C(T,L)\delta^2/3}{\delta/3-\delta^2/3}2I(\tilde{\varphi}_\delta)=0$. i.e. $\lim_{\delta\to0}E_3=0$.

Meanwhile, Let $M=\partial_t\left(\varphi_\delta-\tilde{\varphi}_\delta\right)-\triangle\left(\varphi_\delta-\tilde{\varphi}_\delta\right)-\left(f(\varphi_\delta)-f(\tilde{\varphi}_\delta)\right)$we can see that

$$\begin{aligned}
E_4 &= \int_0^T\int_0^L \left|\frac{\left(\partial_t\tilde{\varphi}_\delta-\triangle\tilde{\varphi}_\delta-f(\tilde{\varphi}_\delta)\right)^2-\left(\partial_t\tilde{\varphi}_\delta-\triangle\tilde{\varphi}_\delta-f(\tilde{\varphi}_\delta)+M\right)^2}{\varphi_\delta}\right|dxdt \\
&\leq 2\int_0^T\int_0^L \left|\frac{\left(\partial_t\tilde{\varphi}_\delta-\triangle\tilde{\varphi}_\delta-f(\tilde{\varphi}_\delta)\right)M}{\varphi_\delta}\right|dxdt+\int_0^T\int_0^L\frac{M^2}{\varphi_\delta}dxdt \\
&:= E_5+E_6.
\end{aligned}$$

From the fact that $\left\|\varphi_\delta-\tilde{\varphi}_\delta\right\|_{W_2^{1,2}}\leq\delta^2/3$ and f is Lipschitz(with constant

σ_3), we have that

$$
\begin{aligned}
& \int_0^T \int_0^L M^2 dxdt \\
\leq\ & 3 \int_0^T \int_0^L \left[\partial_t \left(\varphi_\delta - \tilde{\varphi}_\delta\right)\right]^2 + \left[\triangle \left(\varphi_\delta - \tilde{\varphi}_\delta\right)\right]^2 dxdt + \left[f(\varphi_\delta) - f(\tilde{\varphi}_\delta)\right]^2 dxdt \\
\leq\ & 3 \left[\|\varphi_\delta - \tilde{\varphi}_\delta\|^2_{W_2^{1,2}} + \sigma_3^2 \|\varphi_\delta - \tilde{\varphi}_\delta\|^2_{W_2^{1,2}}\right] \\
\leq\ & 3(i + \sigma_3^2)\delta^4/9 = C\delta^4.
\end{aligned}
$$

Therefore, $\lim_{\delta\to 0} E_6 \leq \lim_{\delta\to 0} \frac{C\delta^4}{\delta/3-\delta^2/3} = 0$. Use Cauchy-Schwarz again and the fact that $\varphi_\delta \geq \delta/3 - \delta^2/3$, we can see that

$$
\begin{aligned}
E_5 \ \leq\ & \frac{2}{\delta/3 - \delta^2/3} \sqrt{\int_0^T \int_0^L \left(\partial_t \tilde{\varphi}_\delta - \triangle \tilde{\varphi}_\delta - f(\tilde{\varphi}_\delta)\right)^2 dxdt} \sqrt{\int_0^T \int_0^L M^2 dxdt} \\
\leq\ & \frac{2C\delta^2}{\delta/3 - \delta^2/3} \sqrt{\int_0^T \int_0^L \left(\partial_t \varphi - \triangle \varphi - f(\tilde{\varphi}_\delta)\right)^2 dxdt}.
\end{aligned}
$$

Since f is bounded and $\varphi \in W_2^{1,2}$, $\sqrt{\int_0^T \int_0^L \left(\partial_t \varphi - \triangle \varphi - f(\tilde{\varphi}_\delta)\right)^2 dxdt}$ is finite. Hence $\lim_{\delta\to 0} E_5 = 0$. i.e. we have proved that

$$\lim_{\delta\to 0} |I(\tilde{\varphi}_\delta) - I(\varphi_\delta)| = 0.$$

Finally, we have shown that

$$\lim_{\delta\to 0} I(\varphi_\delta) = I(\varphi).$$

$\square$

Now we can finally prove the lower bound in the LDP.

Proposition 4.2.4. *Let O be an open set in the space $C_\zeta^+([0,T]\times[0,L])$ when it is equipped with the sup-norm topology. Then we have*

$$\liminf_{\epsilon\to 0} \epsilon^2 \ln P^\epsilon(O) \geq -\inf_{\varphi\in O} I(\varphi). \tag{4.18}$$

Proof. (4.18) is equivalent to

$$\liminf_{\epsilon \to 0} \epsilon^2 \ln P^\epsilon (O) \geq -I(\varphi), \tag{4.19}$$

for any $\varphi \in O$. In fact, we only need to prove (4.19) for any $\varphi \in O$ such that $I(\varphi) = s < \infty$. Since for those φ such that $I(\varphi) = \infty$, (4.19) is obviously true.

From the second statement of lemma 4.2.3, we could see that for any such φ, there exists smooth and strictly positive functions φ_δ, positive numbers δ_1 and $\delta_1^{'}$, such that

$$\begin{aligned} \liminf_{\epsilon \to 0} \epsilon^2 \ln P^\epsilon (O) &\geq \liminf_{\epsilon \to 0} \epsilon^2 \ln P \left(\|u - \varphi\|_{\sup} < \delta \right) \\ &\geq \liminf_{\epsilon \to 0} \epsilon^2 \ln P \left(\|u - \varphi_\delta\|_{\sup} < \delta^{'} \right), \end{aligned}$$

for $0 < \delta < \delta_1$ and $0 < \delta^{'} < \delta_1^{'}$.

By lemma 4.2.2, we have that

$$\liminf_{\epsilon \to 0} \epsilon^2 \ln P \left(\|u - \varphi_\delta\|_{\sup} < \delta^{'} \right) \geq -I(\varphi_\delta) - \frac{1}{2} C_1(\varphi_\delta)\delta^{'} - \frac{1}{2} C(\delta^{'})^2.$$

Letting $\delta^{'}$ go to 0, we can get

$$\liminf_{\epsilon \to 0} \epsilon^2 \ln P \left(\|u - \varphi\|_{\sup} < \delta \right) \geq -I(\varphi_\delta).$$

Finally, let δ go to 0 and by the last statement of lemma 4.2.3, we have

$$\liminf_{\epsilon \to 0} \epsilon^2 \ln P^\epsilon (O) \geq -I(\varphi).$$

And hence the lower bound in the LDP is true. □

Proposition 4.2.5. *Suppose that for any $\phi \in C_\zeta^+ ([0, T] \times [0, L])$, the rate function $I(\phi)$ is defined as in* (4.2). *Then for each $s \geq 0$, the level set of the rate function*

$$\Phi(s) = \left\{ \phi \in C_\zeta^+ ([0, T] \times [0, L]) : I(\phi) \leq s \right\}$$

is compact in $C_\zeta^+ ([0, T] \times [0, L])$.

Proof. From the lower bound of the LDP and the exponential tightness result, the conclusion is true. See [1]. $\square$

4.3 Upper Bound

We will start our proof on the upper bound in LDP by a lemma on the upper bound estimate of probability of open balls in the space $C_\zeta^+([0,T]\times[0,L])$.

Lemma 4.3.1. *Suppose that $u^\epsilon(t,x)$ satisfies* (4.1) *with Neumann boundary condition, where the function f in* (4.1) *satisfies condition $H1-H4$. Let the rate function I be defined as in (4.2). Also suppose that $\|\varphi\|_{\sup}\le N$ for some constant N and $\varphi(t,x)\in C_\zeta^+([0,T]\times[0,L])\cap W_2^{1,2}$ satisfying Neumann boundary condition. Then for any $1>\delta>0$, there exists $0<\varsigma<\delta$ such that*

$$\limsup_{\epsilon\to 0}\epsilon^2\ln P\left(\|u^\epsilon-\varphi\|_{\sup}<\varsigma\right)\le -I(\varphi)+\delta+\varsigma H(\varphi)+C_1\varsigma+C_2\varsigma^2. \quad (4.20)$$

where $C_1=\hat{F}L+\frac{1}{2}\sigma_1 TL$ and $C_2=\sigma_2^2TL$ are constants, and $H(\varphi)=H_1(\varphi)+H_2(\varphi)+I_1(\varphi)$, with

$$H_1(\varphi)=\int_0^T\int_0^L\left|\partial_t\hat{f}(\varphi)+\partial_{xx}\hat{f}(\varphi)\right|dxdt,$$

$$H_2(\varphi)=\sigma_2^2\int_0^T\int_0^L\varphi(t,x)dxdt,$$

$$I_1(\varphi)=\frac{1}{2}\int_0^T\int_0^L\frac{(\partial_t\varphi-\triangle\varphi)^2\mathbf{1}_{\{\varphi>0\}}}{\varphi}dxdt.$$

Proof. We shall use a Girsanov change of measure. Define a new measure Q on $(\Omega,\mathscr{F})$ by

$$\begin{aligned}\frac{dQ}{dP} &:= \mathcal{E}_T \quad (4.21)\\ &= \exp\left\{\int_0^T\int_0^L-\frac{1}{\epsilon}\sqrt{u^\epsilon}\hat{f}(u^\epsilon)W(dxdt)-\frac{1}{2}\int_0^T\int_0^L\frac{1}{\epsilon^2}u^\epsilon\hat{f}^2(u^\epsilon)dxdt\right\}.\end{aligned}$$

The process $\mathcal{E}_t$ is a non-negative local martingale and hence a supermartingale. Therefore to check that it is a true martingale we only need to check that $E^P[\mathcal{E}_t] \geq 1$. We could define a stopping time

$$\tau_N := \inf\left\{t : \int_0^t \int_0^L \frac{1}{\epsilon^2} u^\epsilon \hat{f}^2(u^\epsilon) dx dt \geq N\right\}.$$

Then for each N, the stopped process $\mathcal{E}_{t\wedge\tau_N}$ is a true martingale. Furthermore we could see that

$$E^P[\mathcal{E}_t] \geq E^P\left[\mathcal{E}_{t\wedge\tau_N}\mathbf{1}_{\{\tau_N \geq t\}}\right] = E^P[\mathcal{E}_{t\wedge\tau_N}] - E^P\left[\mathcal{E}_{t\wedge\tau_N}\mathbf{1}_{\{\tau_N < t\}}\right].$$

Since $\lim_{N\to\infty} E^P\left[\mathcal{E}_{t\wedge\tau_N}\mathbf{1}_{\{\tau_N<t\}}\right] = \lim_{N\to\infty} Q(\tau_N < t) = 0$, we could see that $E^P[\mathcal{E}_t] \geq 1$ and hence $\mathcal{E}_t$ is a true martingale. Therefore, Q is a properly defined Girsanov change of measure.

Then define a new noise process $\widetilde{W}$ based on $[0,T]\times[0,L]$ such that

$$\widetilde{W}(B) = W(B) + \int_0^T \int_0^L \mathbf{1}_{\{B\}} \frac{1}{\epsilon}\sqrt{u^\epsilon}\hat{f}(u^\epsilon) dx dt, \tag{4.22}$$

for any $B \in \mathscr{B}([0,T]\times[0,L])$. Then we could check by Girsanov theorem that $\widetilde{W}$ is a Q-Brownian sheet and under Q, u^ϵ satisfies the following SPDE with Neumann boundary condition

$$\begin{aligned} \partial_t u^\epsilon(t,x) &= \triangle u^\epsilon(t,x) + \epsilon\sqrt{u^\epsilon(t,x)}\ddot{\widetilde{W}}_{tx}, \\ u^\epsilon(0,x) &= \zeta(x). \end{aligned} \tag{4.23}$$

For any positive number δ, define set $A := \{\|u^\epsilon - \varphi\|_{\sup} < \delta\}$. Then we have

$$\begin{aligned} & P(A) \\ = \;& E^Q\{\mathbf{1}_{\{A\}}\exp[\frac{1}{\epsilon}\int_0^T\int_0^L \sqrt{u^\epsilon}\hat{f}(u^\epsilon)\widetilde{W}(dxdt) - \frac{1}{2\epsilon^2}\int_0^T\int_0^L u^\epsilon\hat{f}^2(u^\epsilon)dxdt]\} \\ = \;& E^Q\{\mathbf{1}_{\{A\}}\exp[\frac{1}{\epsilon^2}\int_0^T\int_0^L \hat{f}(\varphi)\epsilon\sqrt{u^\epsilon}\widetilde{W}(dxdt) - \frac{1}{2\epsilon^2}\int_0^T\int_0^L u^\epsilon\hat{f}^2(u^\epsilon)dxdt + M_T^\epsilon]\}, \end{aligned} \tag{4.24}$$

where

$$M_T^\epsilon = \frac{1}{\epsilon^2}\int_0^T\int_0^L (\hat{f}(u) - \hat{f}(\varphi))\epsilon\sqrt{u^\epsilon}\widetilde{W}(dxdt).$$

By condition $H3$ that f satisfies, we can see that on the set A,

$$-\int_0^T\int_0^L u^\epsilon \hat{f}^2(u^\epsilon)dxdt \le -\int_0^T\int_0^L \varphi\hat{f}^2(\varphi)dxdt + \sigma_1 TL\delta. \tag{4.25}$$

Meanwhile, since $\varphi \in W_2^{1,2}\cap C_\zeta^+([0,T]\times[0,L])$, there is a sequence of smooth functions $\{\varphi_n(t,x)\}$ in $C_\zeta^+([0,T]\times[0,L])$, such that

$$\lim_{n\to\infty}\|\varphi_n(t,x)-\varphi(t,x)\|_{W_2^{1,2}} = 0.$$

Then for any of those smooth function φ_n, since $\hat{f}$ is twice differentiable, and under probability measure Q, u^ϵ satisfies SPDE (4.23), we could see that

$$\begin{aligned} & \int_0^T\int_0^L \hat{f}(\varphi_n(t,x))\epsilon\sqrt{u^\epsilon(t,x)}\widetilde{W}(dxdt) \qquad (4.26)\\ = & \int_0^L \left[\hat{f}(\varphi_n(T,x))u^\epsilon(T,x) - \hat{f}(\varphi_n(0,x))u^\epsilon(0,x)\right]dx \\ & -\int_0^T\int_0^L u^\epsilon(t,x)\left[\partial_t\hat{f}(\varphi_n(t,x)) + \partial_{xx}\hat{f}(\varphi_n(t,x))\right]dxdt. \end{aligned}$$

It is not difficult to see that

$$\begin{aligned} & \int_0^T\int_0^L \left(\hat{f}\left(\varphi_n(t,x)\right) - \hat{f}\left(\varphi(t,x)\right)\right)^2\epsilon^2 u^\epsilon(t,x)dxdt \qquad (4.27)\\ \le & \int_0^T\int_0^L \sigma_2^2\left(\varphi_n(t,x)-\varphi(t,x)\right)^2\epsilon^2 u^\epsilon(t,x)dxdt. \end{aligned}$$

Since $E^Q\left[\int_0^T\int_0^L \left(u^\epsilon(t,x)\right)^2 dxdt\right] < \infty$ and $\|\varphi\|_{\sup} \le N$ by Cauchy-Schwarz we could see that Q a.s.

$$\lim_{n\to\infty}\int_0^T\int_0^L \hat{f}(\varphi_n(t,x))\epsilon\sqrt{u^\epsilon(t,x)}\widetilde{W}(dxdt) = \int_0^T\int_0^L \hat{f}(\varphi(t,x))\epsilon\sqrt{u^\epsilon(t,x)}\widetilde{W}(dxdt).$$

Similarly we could let $n\to\infty$ on the other side of equation(4.26) and see that for φ, it is true that

$$
\begin{aligned}
&\int_0^T \int_0^L \hat{f}(\varphi(t,x))\epsilon\sqrt{u^\epsilon(t,x)}\widetilde{W}(dxdt) \\
= &\int_0^L \left[\hat{f}(\varphi(T,x))u^\epsilon(T,x) - \hat{f}(\varphi(0,x))u^\epsilon(0,x)\right] dx \\
&-\int_0^T \int_0^L u^\epsilon(t,x) \left[\partial_t \hat{f}(\varphi(t,x)) + \partial_{xx}\hat{f}(\varphi(t,x))\right] dxdt.
\end{aligned}
$$

Also, since we assumed that φ satisfies Neumann boundary condition, from integration by parts formula we have that

$$
\begin{aligned}
&\int_0^T \int_0^L \varphi \left[\partial_t \hat{f}(\varphi) + \partial_{xx}\hat{f}(\varphi)\right] dxdt \\
= &\int_0^L \left[\varphi(T,x)\hat{f}(\varphi(T,x)) - \varphi(0,x)\hat{f}(\varphi(0,x))\right] dx \\
&-\int_0^T \int_0^L \hat{f}(\varphi)\left[\partial_t \varphi - \partial_{xx}\varphi\right] dxdt.
\end{aligned}
$$

Then, combine the above two equalities and we could see that

$$
\begin{aligned}
&\int_0^T \int_0^L \hat{f}(\varphi(t,x))\epsilon\sqrt{u^\epsilon(t,x)}\widetilde{W}(dxdt) \qquad (4.28)\\
= &\int_0^T \int_0^L \hat{f}(\varphi(t,x))\epsilon\sqrt{u^\epsilon(t,x)}\widetilde{W}(dxdt) + \int_0^T \int_0^L \varphi \left[\partial_t \hat{f}(\varphi) + \partial_{xx}\hat{f}(\varphi)\right] dxdt \\
&-\int_0^T \int_0^L \varphi \left[\partial_t \hat{f}(\varphi) + \partial_{xx}\hat{f}(\varphi)\right] dxdt \\
= &\int_0^L \hat{f}(\varphi(T,x))\left[u^\epsilon(T,x) - \varphi(T,x)\right] dx + \int_0^T \int_0^L (\varphi - u^\epsilon)(\partial_t \hat{f}(\varphi) + \partial_{xx}\hat{f}(\varphi))dxdt \\
&+\int_0^T \int_0^L \hat{f}(\varphi)(\partial_t \varphi - \triangle\varphi)dxdt.
\end{aligned}
$$

Then we could conclude that on the set A,

$$
\int_0^T \int_0^L \hat{f}(\varphi)\epsilon\sqrt{u^\epsilon}\widetilde{W}(dxdt) \le \int_0^T \int_0^L \hat{f}(\varphi)(\partial_t \varphi - \triangle\varphi)dxdt + \delta\hat{F}L + \delta H_1(\varphi), \qquad (4.29)
$$

where

$$H_1(\varphi) = \int_0^T \int_0^L \left| \partial_t \hat{f}(\varphi) + \partial_{xx} \hat{f}(\varphi) \right| dxdt.$$

Put (4.29) and (4.25) in (4.24), we could see that

$$\begin{aligned} P(A) \leq\ & E^Q \left\{ \mathbf{1}_{\{A\}} \exp\left(M_T^\epsilon\right) \right\} \\ & \exp\left\{ -\frac{1}{\epsilon^2} \int_0^T \int_0^L \left(-\hat{f}(\varphi)(\partial_t \varphi - \triangle\varphi) + \frac{1}{2}\varphi \hat{f}^2(\varphi) \right) dxdt \right\} \\ & \exp\left\{ \frac{\delta}{\epsilon^2} \left(C_1 + H_1(\varphi) \right) \right\}, \end{aligned} \tag{4.30}$$

where $C_1 = \hat{F}L + \frac{1}{2}\sigma_1 TL$.

For any positive numbers p and q such that $\frac{1}{p} + \frac{1}{q} = 1$, using Hölder's inequality twice and the fact that $\exp\left\{2qM_t^\epsilon - 2q^2\langle M^\epsilon\rangle_t\right\}$ is a supermartingale, we could see that

$$\begin{aligned} & E^Q \left\{ \mathbf{1}_{\{A\}} \exp\left(M_T^\epsilon\right) \right\} \\ \leq\ & \left\{ E^Q \left[\mathbf{1}_{\{A\}} \right] \right\}^{1/p} \left\{ E^Q \left[\mathbf{1}_{\{A\}} \exp\left(qM_T^\epsilon\right) \right] \right\}^{1/q} \\ =\ & \left\{ Q(A) \right\}^{1/p} \left\{ E^Q \left[\mathbf{1}_{\{A\}} \exp\left(qM_T^\epsilon - q^2\langle M^\epsilon\rangle_T\right) \exp\left(q^2\langle M^\epsilon\rangle_T\right) \right] \right\}^{1/q} \\ \leq\ & \left\{ Q(A) \right\}^{1/p} \left\{ \left\{ E^Q \left[\exp\left(2qM_T^\epsilon - 2q^2\langle M^\epsilon\rangle_T\right) \right] \right\}^{1/2} \left\{ E^Q \left[\mathbf{1}_{\{A\}} \exp\left(2q^2\langle M^\epsilon\rangle_T\right) \right] \right\}^{1/2} \right\}^{1/q} \\ \leq\ & \left\{ Q(A) \right\}^{1/p} \left\{ E^Q \left[\mathbf{1}_{\{A\}} \exp\left(2q^2\langle M^\epsilon\rangle_T\right) \right] \right\}^{1/2q}. \end{aligned} \tag{4.31}$$

From the Lipschitz condition on $\hat{f}$ we could see that on set A,

$$\begin{aligned} \langle M^\epsilon\rangle_T &= \frac{1}{\epsilon^2} \int_0^T \int_0^L \left(\hat{f}(u^\epsilon) - \hat{f}(\varphi) \right)^2 u^\epsilon dxdt \\ &= \frac{1}{\epsilon^2} \int_0^T \int_0^L \left(\hat{f}(u^\epsilon) - \hat{f}(\varphi) \right)^2 \varphi + \left(\hat{f}(u^\epsilon) - \hat{f}(\varphi) \right)^2 \left(u^\epsilon - \varphi\right) dxdt \\ &\leq \frac{1}{\epsilon^2} \left(\sigma_2^2 \|\varphi\|_{L^1} \delta^2 + \sigma_2^2 TL\delta^3 \right). \end{aligned}$$

Using this estimate in (4.31) and taking $p = \frac{1}{1-\delta} > 1$, $q = \frac{1}{\delta} > 1$, we will get

$$E^Q \left\{ \mathbf{1}_{\{A\}} \exp\left(M_T^\epsilon\right) \right\} \leq \left\{ Q(A) \right\}^{1-\delta} \exp\left\{ \frac{1}{\epsilon^2} \left(H_2(\varphi)\delta + C_2\delta^2 \right) \right\}, \tag{4.32}$$

where $H_2(\varphi) = \sigma_2^2 \|\varphi\|_{L^1}$ and $C_2 = \sigma_2^2 TL$.

From (4.32) and (4.30), we could conclude that for any $0 < \delta < 1$, we have

$$\begin{aligned} & \limsup_{\epsilon \to 0} \epsilon^2 \ln P(A) \qquad (4.33) \\ \leq \; & \limsup_{\epsilon \to 0} \epsilon^2 \ln Q(A) - \delta \limsup_{\epsilon \to 0} \epsilon^2 \ln Q(A) \\ & - \frac{1}{2} \left\{ \int_0^T \int_0^L \varphi \hat{f}^2(\varphi) - 2\hat{f}(\varphi)(\partial_t \varphi - \triangle \varphi) dx dt \right\} \\ & + \delta \{C_1 + H_1(\varphi) + H_2(\varphi)\} + \delta^2 C_2. \end{aligned}$$

Since A is an open set in the sup-norm topology and under Q, u^ϵ satisfies (4.23), we could use the lower bound in the LDP for Super-Brownian motion we proved before to see that

$$-\delta \limsup_{\epsilon \to 0} \epsilon^2 \ln Q(A) \leq \delta \inf_{\phi \in A} I_1(\phi) \leq \delta I_1(\varphi), \tag{4.34}$$

where

$$I_1(\varphi) = \frac{1}{2} \int_0^T \int_0^L \frac{(\partial_t \varphi - \triangle \varphi)^2 \mathbf{1}_{\{\varphi > 0\}}}{\varphi} dx dt.$$

Also, as we proved in the previous chapter that I_1 is a good rate function for Super-Brownian motion, it is easy to see that for any given $W_2^{1,2}$ function φ, for any $0 < \delta < 1$, there exists $\varsigma > 0$, such that

$$\limsup_{\epsilon \to 0} \epsilon^2 \ln Q \left(\|u^\epsilon - \varphi\|_{\sup} \leq \varsigma \right) \leq -I_1(\varphi) + \delta. \tag{4.35}$$

Since $\varphi \in W_2^{1,2}$, then on the level set $\{(t,x) : \varphi(t,x) = 0\}$, $\partial_t \varphi(t,x) = \triangle \varphi(t,x) = 0$. So

$$\int_0^T \int_0^L \varphi \hat{f}^2(\varphi) - 2\hat{f}(\varphi)(\partial_t \varphi - \triangle \varphi) dx dt = \int_0^T \int_0^L \left(\varphi \hat{f}^2(\varphi) - 2\hat{f}(\varphi)(\partial_t \varphi - \triangle \varphi) \right) \mathbf{1}_{\{\varphi > 0\}} dx dt.$$

Therefore, we could see that

$$I_1(\varphi) - \frac{1}{2} \left\{ \int_0^T \int_0^L \varphi \hat{f}^2(\varphi) - 2\hat{f}(\varphi)(\partial_t \varphi - \triangle \varphi) dx dt \right\} = I(\varphi).$$

Hence, by putting (4.34) and (4.35) into (4.33), we could see that for any given $W_2^{1,2}$ function φ fixed, for any $0<\delta<1$, there exists $\varsigma>0$, such that

$$\limsup_{\epsilon\to 0}\epsilon^2\ln P\left(\|u^\epsilon-\varphi\|_{\sup}<\varsigma\right)\le -I(\varphi)+\delta+\varsigma H(\varphi)+C_1\varsigma+C_2\varsigma^2. \quad (4.36)$$

where $H(\varphi)=H_1(\varphi)+H_2(\varphi)+I_1(\varphi)$. It is obvious that we could take $\varsigma<\delta$ and (4.36) would still be true. □

Remark 4.3.2. It is not difficult to see that if the LDP result in [20] could be checked when the Super-Brownian motion is considered with periodic boundary conditions, this lemma would be true for u^ϵ satisfies (4.1) with periodic boundary conditions. For the case when u^ϵ satisfies (4.1) with Dirichlet boundary condition, the problem with this method is in (4.28), there will be boundary terms involving $\partial_x\varphi-\partial_x u^\epsilon$, which should not obviously be of the scale δ, and hence there is problems in deriving the corresponding version of (4.29). These are the reasons why we only managed to prove the upper bound in the LDP for u^ϵ satisfies (4.1) with Neumann boundary conditions.

Lemma 4.3.3. *Suppose that $u^\epsilon(t,x)$ satisfies* (4.1) *with Neumann boundary condition, where the function f in* (4.1) *satisfies condition $H4$. Let the rate function I_1 be the rate function for super Brownian motion. Also suppose that $\varphi(t,x)\in C_\varsigma^+([0,T]\times[0,L])$, $\|\varphi\|_{\sup}\le N$, and either $\varphi(t,x)\notin W_2^{1,2}$, or $\varphi(t,x)$ does not satisfy Neumann boundary condition. Then for any $\delta>0$, for any $p>0$, $q>0$ such that $\frac{1}{p}+\frac{1}{q}=1$*

$$\begin{aligned}&\limsup_{\epsilon\to 0}\epsilon^2\ln P\left(\|u^\epsilon-\varphi\|_{\sup}<\delta\right)\\ \le\ & -\left(\frac{1}{p}+\frac{1}{2q}\right)\inf_{\phi\in B(\varphi,\delta)}I_1(\phi)+\left(q-\frac{1}{2}\right)(N+\delta)\hat{F}^2TL.\end{aligned}$$

Proof. Same as in the previous lemma, we define the new measure Q on $(\Omega,\mathscr{F})$ by

$$\begin{aligned}\frac{dQ}{dP} &:= \exp\left\{M_T-\frac{1}{2}\langle M\rangle_T\right\} \qquad (4.37)\\ &= \exp\left\{\int_0^T\int_0^L-\frac{1}{\epsilon}\sqrt{u^\epsilon}\hat{f}(u^\epsilon)W(dxdt)-\frac{1}{2}\int_0^T\int_0^L\frac{1}{\epsilon^2}u^\epsilon\hat{f}^2(u^\epsilon)dxdt\right\}.\end{aligned}$$

Also, define set $A:=\left\{\|u^\epsilon(t,x)-\varphi(t,x)\|_{\sup}<\delta\right\}$. It is easy to see that for

$u^\epsilon \in A$, $\|\varphi\|_{\sup} \leq N$ and $\hat{f}$ satisfies condition $H4$

$$\langle M\rangle_T = \frac{1}{\epsilon^2}\int_0^T\int_0^L u^\epsilon \hat{f}^2(u^\epsilon)dxdt \leq \frac{1}{\epsilon^2}(N+\delta)\hat{F}^2TL.$$

Then use the same method as we used in deriving (4.31), we have for any positive number p and q such that $\frac{1}{p}+\frac{1}{q}=1$

$$\begin{aligned}
& P\left(A\right) \qquad\qquad (4.38)\\
= \;& E^Q\left[\mathbf{1}_{\{A\}}\exp\left(M_T - \frac{1}{2}\langle M\rangle_T\right)\right]\\
\leq \;& \{Q\left(A\right)\}^{1/p}\left\{E^Q\left[\mathbf{1}_{\{A\}}\exp\left(qM_T - \frac{q}{2}\langle M\rangle_T\right)\right]\right\}^{1/q}\\
= \;& \{Q\left(A\right)\}^{1/p}\left\{E^Q\left[\mathbf{1}_{\{A\}}\exp\left(qM_T - q^2\langle M\rangle_T\right)\exp\left(q^2\langle M\rangle_T - \frac{q}{2}\langle M\rangle_T\right)\right]\right\}^{1/q}\\
\leq \;& \{Q\left(A\right)\}^{1/p}\left\{\left(E^Q\left[\exp\left(2qM_T - 2q^2\langle M\rangle_T\right)\right]\right)^{1/2}\left(E^Q\left[\mathbf{1}_{\{A\}}\exp\left(2q^2\langle M\rangle_T - q\langle M\rangle_T\right)\right]\right)^{1/2}\right\}^{1/q}\\
\leq \;& \{Q\left(A\right)\}^{1/p}\left\{E^Q\left[\mathbf{1}_{\{A\}}\exp\left(2q^2\langle M\rangle_T - q\langle M\rangle_T\right)\right]\right\}^{1/2q}\\
\leq \;& \{Q\left(A\right)\}^{1/p+1/2q}\exp\left(\frac{1}{\epsilon^2}(q-\frac{1}{2})(N+\delta)\hat{F}^2TL\right)
\end{aligned}$$

Therefore, since under Q u^ϵ satisfies equation (4.23) with Neumann boundary conditions, we could use the upper bound in large deviation for super Brownian motion to see that

$$\begin{aligned}
& \limsup_{\epsilon\to 0}\epsilon^2\ln P\left(\|u^\epsilon - \varphi\|_{\sup} < \delta\right) \qquad\qquad (4.39)\\
\leq \;& \left(\frac{1}{p}+\frac{1}{2q}\right)\limsup_{\epsilon\to 0}\epsilon^2\ln Q\left(\|u^\epsilon - \varphi\|_{\sup} < \delta\right) + \left(q-\frac{1}{2}\right)\left(N+\delta\right)\hat{F}^2TL\\
\leq \;& -\left(\frac{1}{p}+\frac{1}{2q}\right)\inf_{\phi\in B(\varphi,\delta)} I_1(\phi) + \left(q-\frac{1}{2}\right)\left(N+\delta\right)\hat{F}^2TL
\end{aligned}$$

as required. □

The next step would be to prove an upper bound for a general compact set C in $C_\zeta^+([0,T]\times[0,L])$. Then since we have already proved an exponential tightness result, these would imply the upper bound in LDP is true for any general closed set.

Lemma 4.3.4. *Suppose that P^ϵ is the probability measure on $C_\zeta^+([0,T]\times[0,L])$ induced by u^ϵ satisfying* (4.1)*. Suppose that the function f in* (4.1) *satisfies condition $H1-H4$. For any function $\phi \in C_\zeta^+([0,T]\times[0,L])$, let the rate function $I(\phi)$ be defined as in* (4.2)*. Then for any compact set $C \subset C_\zeta^+([0,T]\times[0,L])$, it is true that*

$$\limsup_{\epsilon\to 0} \epsilon^2 \ln P^\epsilon(C) \le -\inf_{\phi\in C} I(\phi).$$

Proof. For any compact set $C \subset C_\zeta^+([0,T]\times[0,L])$, it can be written as

$$C = \left(C \cap \left\{\phi : \|\phi\|_{\sup} \le N\right\}\right) \cup \left(C \cap \left\{\phi : \|\phi\|_{\sup} > N\right\}\right) := C_N^1 \cup C_N^2. \tag{4.40}$$

Obviously it is true that $P^\epsilon\left(C_N^2\right) \le P^\epsilon\left(\left\{\phi : \|\phi\|_{\sup} > N\right\}\right)$. Then from the exponential tightness result, we know that for any $K \in R$, there exists N, such that

$$\limsup_{\epsilon\to 0} \epsilon^2 \ln P^\epsilon\left(C_N^2\right) < -K.$$

If we could prove that

$$\limsup_{\epsilon\to 0} \epsilon^2 \ln P^\epsilon\left(C_N^1\right) \le -\inf_{\phi\in C_N^1} I(\phi). \tag{4.41}$$

Then we could take $K > \inf_{\phi\in C_N^1} I(\phi)$, and see that

$$\begin{aligned}
\limsup_{\epsilon\to 0} \epsilon^2 \ln P^\epsilon(C) &= \max\left(\limsup_{\epsilon\to 0} \epsilon^2 \ln P^\epsilon\left(C_N^1\right), \limsup_{\epsilon\to 0} \epsilon^2 \ln P^\epsilon\left(C_N^2\right)\right) \\
&\le \max\left(-\inf_{\phi\in C_N^1} I(\phi), -K\right) \\
&= -\inf_{\phi\in C_N^1} I(\phi) \le -\inf_{\phi\in C} I(\phi).
\end{aligned}$$

i.e. the result of this lemma is true. So the rest of this proof will be showing that (4.41) is true.

For any given $\delta > 0$, if $\varphi \in C_N^1 \cap W_2^{1,2}$ satisfying Neumann boundary condition, by Lemma 4.3.1, there exists $\varsigma < \delta$, such that

$$\limsup_{\epsilon\to 0} \epsilon^2 \ln P\left(B\left(\varphi,\varsigma\right)\right) \le -I(\varphi) + \delta + \varsigma H(\varphi) + C_1\varsigma + C_2\varsigma^2. \tag{4.42}$$

Letting $\delta \to 0$, we could see that

$$\lim_{\delta\to 0}\limsup_{\epsilon\to 0}\epsilon^2 \ln P\left(B\left(\varphi,\varsigma\right)\right) \le -I(\varphi). \tag{4.43}$$

On the other hand, if $\varphi \in C_N^1 \cap \left(W_2^{1,2}\right)^c$, or $\varphi \in W_2^{1,2}$ but it does not satisfy Neumann boundary condition, by Lemma 4.3.3, for any $p > 0$, $q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$

$$\limsup_{\epsilon\to 0}\epsilon^2 \ln P\left(B\left(\varphi,\delta\right)\right) \le -\left(\frac{1}{p}+\frac{1}{2q}\right)\inf_{\phi\in B(\varphi,\delta)} I_1(\phi) + \left(q-\frac{1}{2}\right)\left(N+\delta\right)\hat{F}^2 TL. \tag{4.44}$$

Again letting $\delta \to 0$, as $\inf_{\phi\in B(\varphi,\delta)} I_1(\phi)$ increases as $\delta \to 0$, and $I_1(\varphi) = \infty$, we see that

$$\lim_{\delta\to 0}\limsup_{\epsilon\to 0}\epsilon^2 \ln P\left(B\left(\varphi,\varsigma\right)\right) = -\infty. \tag{4.45}$$

Now for any δ, we could cover C_N^1 by open balls of suitable radius (i.e for non $W_2^{1,2}$ function, of radius δ, for $W_2^{1,2}$ function, of radius $\varsigma < \delta$ decided by δ and the function itself) around all functions in it. Since it is a compact set, there exists a finite cover of it, i.e.

$$C_N^1 \subset \cup_{\phi_i\in C_N^1} B(\phi_i,\varsigma_i) \subset \cup_{i\in\{1,2,\dots,m\}} B(\phi_i,\varsigma_i),$$

Then we have

$$\limsup_{\epsilon\to 0}\epsilon^2 \ln P^\epsilon\left(C_N^1\right) \le \max_{i\in\{1,2,\dots,m\}}\left(\limsup_{\epsilon\to 0}\epsilon^2 \ln P^\epsilon\left(B(\phi_i,\varsigma_i)\right)\right) \tag{4.46}$$

Now let $\delta \to 0$, by (4.43) and (4.45), we could see that

$$\limsup_{\epsilon\to 0}\epsilon^2 \ln P^\epsilon\left(C_N^1\right) \le \sup_{\phi_i\in C_N^1}\left(\lim_{\delta\to 0}\limsup_{\epsilon\to 0}\epsilon^2 \ln P^\epsilon\left(B(\phi_i,\varsigma_i)\right)\right) \le -\inf_{\phi\in C_N^1} I(\phi) \tag{4.47}$$

i.e. (4.41) is true and hence the conclusion of this lemma. □

Proposition 4.3.5. *Suppose that P^ϵ is the probability measure on $C_\zeta^+\left([0,T]\times[0,L]\right)$ induced by u^ϵ satisfying* (4.1). *The function f in* (4.1) *satisfies condition $H1 - H4$. For any function $\phi \in C_\zeta^+\left([0,T]\times[0,L]\right)$, the rate function $I(\phi)$is defined as in* (4.2). *Then for any closed set $C \subset C_\zeta^+\left([0,T]\times[0,L]\right)$, it is* *true that*

$$\limsup_{\epsilon \to 0} \epsilon^2 \ln P^\epsilon(C) \leq - \inf_{\phi \in C} I(\phi).$$

Proof. From the result in Lemma 4.1.1 and Lemma 4.3.4, we know that the conclusion of this proposition is true, see Lemma 1.2.18 in [1]. □

Chapter 5

Applications of LDP for Stochastic Reaction-Diffusion Equation

In the previous two chapters, we studied the large deviation principle for the probability measures on $C_{\zeta}^{+}([0,T]\times[0,L])$ induced by Super-Brownian motion

$$\begin{aligned}\partial_t u^\epsilon(t,x) &= \partial_{xx}u^\epsilon(t,x)+\epsilon\sqrt{u^\epsilon(t,x)}\ddot{W}_{tx}, \\ u^\epsilon(0,x) &= \zeta(x),\end{aligned} \tag{5.1}$$

and a stochastic reaction-diffusion equation

$$\begin{aligned}\partial_t u^\epsilon(t,x) &= \partial_{xx}u^\epsilon(t,x)+f(u(t,x))+\epsilon\sqrt{u^\epsilon(t,x)}\ddot{W}_{tx}, \\ u^\epsilon(0,x) &= \zeta(x).\end{aligned} \tag{5.2}$$

Here both equations are considered for $\{(t,x)\in[0,T]\times[0,L]\}$ and W is a time-space Brownian sheet defined on a probability space $(\Omega,\mathscr{F},P)$. Also both equations can be considered with one of the following three boundary conditions: periodic, Dirichlet, or Neumann.The initial condition ζ satisfies $\zeta(x)\geq 0$, for all $x\in[0,L]$ and is a continuous deterministic function on $[0,L]$ which satisfies the boundary condition that the equation is considered with. In both of the two cases, the space-time white noise W can kill off the solution so that the events $\{u^\epsilon(T,x)=0,\forall x\in[0,L]\}$ and

$\{u^\epsilon(T,x) \leq \delta, \forall x \in [0,L]\}$ for some small $\delta > 0$ would be interesting to calculate the probability of. In this chapter, we will try to apply the large deviation principle we proved in the two previous chapters to explore these probabilities in the limit of small noise. In the first half, we give some formal arguments that lead to a conjecture of the probabilities of the above events. Some parts of these arguments can be made rigorous. In the case when $f \equiv 0$, we can use other methods to verify our conjecture and indeed obtain stronger results. In the second half, we will try to explore a method to calculate $P\{(u^\epsilon(t,x) \in A) \,|\, (u^\epsilon(T,x) = 0, \forall x \in [0,L])\}$ for certain set of continuous functions A. We will use a Girsanov change of measure to transform this conditional probability to a unconditional one. Under the new measure, $u^\epsilon(t,x)$ would satisfy a different SPDE.

5.1 Calculation of $P\{u^\epsilon(T,x) = 0, \forall x \in [0,L]\}$

Suppose that $u^\epsilon(t,x)$ satisfies (5.2) with one of the three possible boundary conditions. We will explore possible methods to calculate $P\{u^\epsilon(T,x) = 0, \forall x \in [0,L]\}$. Then the case when $u^\epsilon(t,x)$ satisfies (5.1) can be seen as a special situation (setting $f(u) \equiv 0$). As previously mentioned, we will try to use the large deviation principle. Although we only rigorously proved the LDP for (5.2) with Neumann boundary condition and $\zeta(x) > 0$, we do believe the LDP would be true for (5.2) with all three different boundary conditions and also when $\zeta(x) \geq 0$. Therefore our investigation will be for (5.2) with any of the three different boundary conditions and $\zeta(x) \geq 0$. Since we consider u^ϵ in $C_\zeta^+([0,T] \times [0,L])$ with the sup-norm topology, $\{u^\epsilon(T,x) = 0, \forall x \in [0,L]\}$ is a closed set whose interior is empty. Therefore we only have the upper bound of $P\{u^\epsilon(T,x) = 0, , \forall x \in [0,L]\}$ in the LDP, i.e.

$$\limsup_{\epsilon \longrightarrow 0} \epsilon^2 \ln P\{u^\epsilon(T,x) = 0, \forall x \in [0,L]\} \leq -\inf_{\phi \in A} I(\phi)$$

where $A = \left\{\phi \in C_\zeta^+([0,T] \times [0,L]) : \phi(T,x) = 0, \forall x \in [0,L]\right\}$ and the rate function is

$$I(\phi) = \begin{cases} \frac{1}{2}\int_0^T \int_0^L \frac{(\partial_t \phi - \triangle \phi - f(\phi))^2 \mathbf{1}_{\{\phi > 0\}}}{\phi} dxdt & \text{if } \phi \in W_2^{1,2}, \text{ satisfies the boundary condition as } u^\epsilon, \\ \infty & \text{otherwise.} \end{cases} \tag{5.3}$$

But this upper bound helps to link the problem of estimating $P\{u^\epsilon(T,x)=0, \forall x\in[0,L]\}$ as $\epsilon\to 0$ and a problem of calculus of variation. So we will first try to explore the calculus of variation problem and then try to see whether our guess for the minimizer of the rate function over a suitable set really gives us useful information about $P\{u^\epsilon(T,x)=0, \forall x\in[0,L]\}$.

5.1.1 Euler-Lagrange equation and a candidate for minimizer

It is easy to see that the minimization problem is

$$\inf_{\phi\in A} I(\phi) = \frac{1}{2}\inf_{\phi\in\mathscr{D}} F(\phi) \tag{5.4}$$

where

$$\begin{aligned} F(\phi) \quad &:= \quad \int_0^T\int_0^L h\left(\phi(t,x),\phi_t(t,x),\phi_{xx}(t,x)\right)dxdt \\ &:= \quad \int_0^T\int_0^L \frac{\left(\phi_t-\phi_{xx}-f(\phi)\right)^2\mathbf{1}_{\{\phi>0\}}}{\phi}dxdt. \end{aligned} \tag{5.5}$$

When (5.2) is considered with periodic boundary condition,

$$\mathscr{D}=\mathscr{D}^P=\left\{\begin{array}{ll}\phi\in W_2^{1,2}\left([0,T]\times[0,L]\right): & \phi\geq 0, \phi(0,x)=\zeta(x), \phi(T,x)\equiv 0, \\ \partial_x\phi(t,0)=\partial_x\phi(t,L), & \phi(t,0)=\phi(t,L), F(\phi)<\infty\end{array}\right\},$$

when it is considered with Dirichlet boundary conditions,

$$\mathscr{D}=\mathscr{D}^D=\left\{\begin{array}{ll}\phi\in W_2^{1,2}\left([0,T]\times[0,L]\right): & \phi\geq 0, \phi(0,x)=\zeta(x), \phi(T,x)\equiv 0, \\ & \phi(t,0)=\phi(t,L)=0, F(\phi)<\infty\end{array}\right\},$$

and when it is considered with Neumann boundary conditions,

$$\mathscr{D}=\mathscr{D}^N=\left\{\begin{array}{ll}\phi\in W_2^{1,2}\left([0,T]\times[0,L]\right): & \phi\geq 0, \phi(0,x)=\zeta(x), \phi(T,x)\equiv 0, \\ & \partial_x\phi(t,0)=\partial_x\phi(t,L)=0, F(\phi)<\infty\end{array}\right\}.$$

We will follow the standard method (like in [17]) to derive the Euler-Lagrange equation for this minimization problem and give our guess for the minimizer.

By definition, $\phi \in \mathscr{D}$ is a minimizer of $F(\phi)$ over $\mathscr{D}$ if and only if

$$F(\phi + v) \geq F(\phi) \textit{ whenever all } \phi + v \in \mathscr{D}.$$

In any of our cases, since $\mathscr{D}$ is not a linear space, we define the following admissible space, in the case of Dirichlet boundary condition,

$$\mathscr{D}_{0,\phi} = \mathscr{D}^D_{0,\phi} = \left\{ \begin{array}{ll} v \in W_2^{1,2}\left([0,T] \times [0,L]\right): & \phi + \epsilon v \geq 0, v(0,x) = 0, v(T,x) \equiv 0, \\ v(t,0) = v(t,L) = 0, & F(\phi + \epsilon v) < \infty \text{ for } \phi \in \mathscr{D} \text{ and } \epsilon \text{ small enough} \end{array} \right\}.$$

And we can see that we could define $\mathscr{D}^P_{0,\phi}$ and $\mathscr{D}^N_{0,\phi}$ when (5.2) is considered with periodic or Neumann boundary conditions respectively. Then for any $\phi \in \mathscr{D}$, $v \in \mathscr{D}_{0,\phi}$, we have $\phi + \epsilon v \in \mathscr{D}$ for ϵ small enough. Next, for any $\phi \in \mathscr{D}$, $v \in \mathscr{D}_{0,\phi}$, define the Gateaux variations $\delta F(\phi; v)$ as

$$\delta F(\phi; v) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[F(\phi + \varepsilon v) - F(\phi)].$$

In our case, we have formally

$$\begin{aligned}
& \delta F(\phi; v) \\
= & \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^T \int_0^L \left[\frac{\left(\phi_t + \epsilon v_t - (\phi_{xx} + \epsilon v_{xx}) - f\left(\phi + \epsilon v\right)\right)^2}{\phi + \epsilon v} - \frac{\left(\phi_t - \phi_{xx} - f\left(\phi\right)\right)^2}{\phi} \right] dxdt \\
= & \lim_{\epsilon \to 0} \int_0^T \int_0^L \frac{1}{\epsilon} \left[\frac{\left(\phi_t + \epsilon v_t - (\phi_{xx} + \epsilon v_{xx}) - f\left(\phi + \epsilon v\right)\right)^2}{\phi + \epsilon v} - \frac{\left(\phi_t - \phi_{xx} - f\left(\phi\right)\right)^2}{\phi + \epsilon v} \right] dxdt \\
& + \lim_{\epsilon \to 0} \int_0^T \int_0^L \frac{1}{\epsilon} \left[\frac{\left(\phi_t - \phi_{xx} - f\left(\phi\right)\right)^2}{\phi + \epsilon v} - \frac{\left(\phi_t - \phi_{xx} - f\left(\phi\right)\right)^2}{\phi} \right] dxdt \\
= & \lim_{\epsilon \to 0} \int_0^T \int_0^L \left[\frac{2\left(\phi_t - \phi_{xx} - f\left(\phi\right)\right)\left(v_t - v_{xx} - \frac{1}{\epsilon}\left(f\left(\phi + \epsilon v\right) - f\left(\phi\right)\right)\right)}{\phi + \epsilon v} \right] dxdt \\
& - \lim_{\epsilon \to 0} \int_0^T \int_0^L \frac{\left(\phi_t - \phi_{xx} - f\left(\phi\right)\right)^2 v}{\phi\left(\phi + \epsilon v\right)} dxdt \\
& + \lim_{\epsilon \to 0} \int_0^T \int_0^L \frac{\epsilon\left(v_t - v_{xx} - \frac{1}{\epsilon}\left(f\left(\phi + \epsilon v\right) - \left(\phi\right)\right)\right)^2}{\phi + \epsilon v} dxdt
\end{aligned}$$

It not obvious that we could pass the limit inside the integral, but for now we assume that we could and then we can see that

$$\delta F(\phi;v)=\int_0^T\int_0^L h_x(\phi,\phi_t,\phi_{xx})v+h_y(\phi,\phi_t,\phi_{xx})v_t+h_z(\phi,\phi_t,\phi_{xx})v_{xx}dxdt$$

where $h(x,y,z)=\frac{(y-z-f(x))^2}{x}$, $h_x(x,y,z)=-\frac{(y-z-f(x))^2}{x^2}-\frac{2f'(x)(y-z-f(x))}{x}$, $h_y(x,y,z)=\frac{2(y-z-f(x))}{x}$ and $h_z(x,y,z)=-\frac{2(y-z-f(x))}{x}$.

For any $\phi\in\mathscr{D}$, the equation $\delta F(\phi;v)=0$ for all $v\in\mathscr{D}_{0,\phi}$ is called the Euler-Lagrange equation. Normally, the Euler-Lagrange equation is only a necessary condition for a minimizer. But in our problem, when $f\equiv 0$, i.e. u^ϵ satisfies (5.1), there is a hope that it could be a sufficient condition as well. We could see that the integrand function h has the following convexity property, for $(x,y,z)\in R^+\times R\times R$ and $(a,b,c)\in R\times R\times R$, such that $x+a>0$

$$h(x+a,y+b,z+c)-h(x,y,z)\geq h_x(x,y,z)a+h_y(x,y,z)b+h_z(x,y,z)c$$

This is simple to prove as by calculation we can see that

$$\begin{aligned}&h(x+a,y+b,z+c)-h(x,y,z)-[h_x(x,y,z)a+h_y(x,y,z)b+h_z(x,y,z)c]\\=\;&\frac{[x(b-c)-a(y-z)]^2}{x^2(x+a)}\geq 0\end{aligned}$$

If we were minimizing F over strictly positive functions, we could continue to see that for $\phi\in\mathscr{D}$, $\phi>0$ if $\delta F(\phi;v)=0$, for all $v\in\mathscr{D}_{0,\phi}$, $\epsilon v+\phi>0$ then

$$\begin{aligned}&F(\phi+\epsilon v)-F(\phi)\\=\;&\int_0^T\int_0^L h(\phi+v,\phi_t+v_t,\phi_{xx}+v_{xx})-h(\phi,\phi_t,\phi_{xx})dxdt\\\geq\;&\int_0^T\int_0^L h_x(\phi,\phi_t,\phi_{xx})v+h_y(\phi,\phi_t,\phi_{xx})v_t+h_z(\phi,\phi_t,\phi_{xx})v_{xx}dxdt\\=\;&\delta F(\phi;v)=0\end{aligned}$$

i.e. Any solution to the Euler-Lagrange equation would be a minimizer. So, although there are still problems about the strict positivity, we still believe the following conjecture should be true for the case $f \equiv 0$.

Conjecture 5.1.1. *If there exists $\phi \in \mathscr{D}$ such that for all $v \in \mathscr{D}_0$, $\delta F(\phi; v) = 0$, then ϕ is a minimizer of F over $\mathscr{D}$.*

As pointed out in many examples in [17], convexity is not the only one condition that would make the Euler-Lagrange equation a sufficient condition for a minimizer. Although we can't see how this would be proved in the case when f is not identically zero, we still believe that this might be true. Next, we will do some simple calculations to get a pair of forward-backward partial differential equations which is a sufficient condition of the Euler-Lagrange equation.

Since $\phi, v \in W_2^{1,2}$ and h,h_x,h_y,h_z are continuous, we could use integration by parts formula and the fact that $v(0,x) = v(T,x) = 0$ for $v \in \mathscr{D}^D_{0,\phi}$ $\mathscr{D}^N_{0,\phi}$, or $\mathscr{D}^P_{0,\phi}$ to see that

$$
\begin{aligned}
& \int_0^T \int_0^L h_y(\phi, \phi_t, \phi_{xx}) v_t dx dt \\
= & \int_0^L \left[h_y(\phi, \phi_t, \phi_{xx}) v|_{t=0}^T - \int_0^T v \left(\frac{\partial}{\partial t} h_y(\phi, \phi_t, \phi_{xx}) \right) dt \right] dx \\
= & -\int_0^T \int_0^L v \left(\frac{\partial}{\partial t} h_y(\phi, \phi_t, \phi_{xx}) \right) dt dx.
\end{aligned}
$$

Similarly, by integration by parts formula, when u^ϵ satisfies (5.2) with Dirichlet boundary conditions,

$$
\begin{aligned}
& \int_0^T \int_0^L h_z(\phi, \phi_t, \phi_{xx}) v_{xx} dx dt \\
= & \int_0^T v_x h_z(\phi, \phi_t, \phi_{xx})|_{x=0}^L dt + \int_0^T \int_0^L v \left(\frac{\partial^2}{\partial x^2} h_z(\phi, \phi_t, \phi_{xx}) \right) dx dt,
\end{aligned}
$$

when u^ϵ satisfies (5.2) with Neumann boundary conditions

$$
\begin{aligned}
& \int_0^T \int_0^L h_z(\phi, \phi_t, \phi_{xx}) v_{xx} dx dt \\
= & -\int_0^T v \left(\frac{\partial}{\partial x} h_z(\phi, \phi_t, \phi_{xx}) \right) |_{x=0}^L dt + \int_0^T \int_0^L v \left(\frac{\partial^2}{\partial x^2} h_z(\phi, \phi_t, \phi_{xx}) \right) dx dt,
\end{aligned}
$$

and when u^ϵ satisfies (5.2) with periodic boundary conditions

$$\int_0^T \int_0^L h_z(\phi, \phi_t, \phi_{xx}) v_{xx} dx dt = \int_0^T \int_0^L v \left(\frac{\partial^2}{\partial x^2} h_z(\phi, \phi_t, \phi_{xx}) \right) dx dt.$$

Therefore,

$$\begin{aligned} \delta F(\phi; v) \quad = \quad & \int_0^T \int_0^L v \left[h_x(\phi, \phi_t, \phi_{xx}) - \frac{\partial}{\partial t} h_y(\phi, \phi_t, \phi_{xx}) + \frac{\partial^2}{\partial x^2} h_z(\phi, \phi_t, \phi_{xx}) \right] dx dt \\ & + \text{boundary terms.} \end{aligned}$$

Then when u^ϵ satisfies (5.2) with Dirichlet boundary conditions, if there exists any $\phi \in \mathscr{D}^D$ satisfies

$$\begin{cases} \frac{\partial}{\partial t} h_y(\phi, \phi_t, \phi_{xx}) = \frac{\partial^2}{\partial x^2} h_z(\phi, \phi_t, \phi_{xx}) + h_x(\phi, \phi_t, \phi_{xx}), \\ h_z(\phi(t,0), \phi_t(t,0), \phi_{xx}(t,0)) = 0 \ \forall t \in [0,T], \\ h_z(\phi(t,L), \phi_t(t,L), \phi_{xx}(t,L)) = 0 \ \forall t \in [0,T], \end{cases} \tag{5.6}$$

then $\delta F(\phi; v) = 0$ for all $v \in \mathscr{D}^D_{0,\phi}$, i.e. ϕ is a minimizer. Similarly, when u^ϵ satisfies (5.2) with Neumann boundary conditions, the sufficient condition would still have the equation in (5.6), but the boundary condition would be

$$\begin{cases} \frac{\partial}{\partial x} h_z(\phi(t,0), \phi_t(t,0), \phi_{xx}(t,0)) = 0 \ \forall t \in [0,T], \\ \frac{\partial}{\partial x} h_z(\phi(t,L), \phi_t(t,L), \phi_{xx}(t,L)) = 0 \ \forall t \in [0,T]. \end{cases} \tag{5.7}$$

And when u^ϵ satisfies (5.2) with periodic boundary conditions, the sufficient condition would be just the equation in (5.6).

Suppose for the moment we consider $\phi(t,x) > 0$ and define

$$\hat{\psi}(t,x) = \frac{\phi_t(t,x) - \phi_{xx}(t,x) - f(\phi(t,x))}{\phi(t,x)},$$

we can see that the equation in (5.6) is equivalent to the following pair of equations

$$\begin{cases} \hat{\psi}_t(t,x) = -\hat{\psi}_{xx}(t,x) - \frac{1}{2}\hat{\psi}^2(t,x) - f(\phi(t,x))\psi(t,x). \\ \phi_t(t,x) = \phi_{xx}(t,x) + f(\phi(t,x)) + \hat{\psi}(t,x)\phi(t,x). \end{cases} \tag{5.8}$$

By letting $\psi(t,x) = -\hat{\psi}(T-t,x)$, we will get our conjecture for a sufficient condition of the minimizer, which is as follows. Any ϕ that satisfies $\phi(0,x) = \zeta(x)$, $\phi(T,x) \equiv 0$, and

$$\begin{cases} \psi_t(t,x) = \psi_{xx}(t,x) - \frac{1}{2}\psi^2(t,x) + f^{'}(\phi(T-t,x))\psi(t,x), \\ \phi_t(t,x) = \phi_{xx}(t,x) - \psi(T-t,x)\phi(t,x) + f(\phi(t,x)), \end{cases} \tag{5.9}$$

where both ϕ and ψ are considered with the same boundary condition that (5.2) is considered with, is a minimizer of the calculus of variation problem (5.4). In this pair of fully-coupled forward and backward PDEs, there are boundary conditions, initial and terminal conditions for ϕ but only boundary conditions for ψ. So we hope there will be some initial condition we could impose on ψ to make sure that ϕ satisfies its terminal condition. Intuitively, that initial condition would be $\psi(0,x) = \infty$, because then and only then, will $\phi(T,x) \equiv 0$. But this condition requires care to interpret, as a limit of increasing initial conditions. Here is one possible approximation scheme to construct suitable solution. We start from function $\psi_0(t,x) \equiv 0$, put it in the ϕ equation in (5.9) with initial condition $\phi_1(0,x) = \zeta(x)$ and boundary conditions to get the function ϕ_1, then use this ϕ_1 in the ψ equation in (5.9) with initial condition $\psi_1(0,x) = 1 \times \theta(x)$ and boundary conditions to get ψ_1. By repeating this procedure we will get a sequence of paired functions $\{\psi_n(t,x), \phi_n(t,x)\}_{n=1}^{\infty}$ that at each n, the paired functions satisfy

$$\begin{cases} \partial_t\phi_n(t,x) = \partial_{xx}\phi_n(t,x) - \psi_{n-1}(T-t,x)\phi_n(t,x) + f(\phi_n(t,x)), \\ \phi_n(0,x) = \zeta(x), \\ \partial_t\psi_n(t,x) = \partial_{xx}\psi_n(t,x) - \frac{1}{2}\psi_n^2(t,x) + f^{'}(\phi_n(T-t,x))\psi_n(t,x), \\ \psi_n(0,x) = n\theta(x), \end{cases} \tag{5.10}$$

where both of them are considered with the same boundary condition as (5.2) and $0 < \theta(x) \leq 1$ is a smooth function satisfies the same boundary condition. Then we hope that there exists functions $\phi, \psi \in W_2^{1,2}$ $\phi(t,x) = \lim_{n\to\infty} \phi_n(t,x)$, $\psi(t,x) = \lim_{n\to\infty} \psi_n(t,x)$, such that the pair satisfies (5.9) and $\phi(T,x) \equiv 0$.

This is a big conjecture to prove. The main difficulty is that this pair is fully coupled. If we ignore the initial and terminal condition and just consider the forward-backword pair equation (5.9), it might possible to prove there exists solution for small T only. But note that when $f \equiv 0$, the equation of ψ will be independent of ϕ. So in the rest of this section, we are going to consider this easier case. First we will use comparison theorems for PDEs to prove that when $f \equiv 0$, $\{\phi_n\}$ as constructed in (5.10) is a decreasing sequence of functions and $\{\psi_n\}$ as constructed in (5.10) is an increasing sequence of functions. Then we will show that there exists function $\phi(t,x) = \lim_{n\to\infty} \phi_n(t,x)$ and $\psi(t,x) = \lim_{n\to\infty} \psi_n(t,x)$. We also claim that the limit function ϕ satisfies $\phi(T,x) \equiv 0$, which can be proved rigorously using Feyman-Kac formula in some special case. Finally we will use a standard martingale approach to calculate $P\{u^\epsilon(T,x) \equiv 0\}$ when u^ϵ satisfies (5.1), which will confirm with the result by putting the function ϕ we constructed above in the rate function of large deviation principle.

First we state here the comparison theorem we are going to use many times below. This can be proved easily using Theorem 8.1.2 from [29].

Lemma 5.1.2. *Let $u \in C^{1,2}((0,T)\times(0,L)) \cap C([0,T]\times[0,L])$. Suppose that it satisfies*

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) + c(t,x)u(t,x) \geq 0, & (t,x) \in (0,T)\times(0,L), \\ u(0,x) \geq 0, & x \in [0,L], \\ \partial_x u(t,0) = 0,\ \partial_x u(t,L) = 0, & t \in [0,T], \end{cases} \tag{5.11}$$

where $c(t,x)$ is a continuous function on $[0,T]\times[0,L]$. Then $u(t,x) \geq 0$, for all $(t,x) \in [0,T]\times[0,L]$. If the boundary condition in (5.11) *is replaced by*

$$u(t,0) \geq 0,\ u(t,L) \geq 0,\ \ t \in [0,T],$$

the same conclusion is valid as well.

Lemma 5.1.3. *Suppose that $\{\psi_n\}$ is a sequence of functions each of them satisfies equation*

$$\begin{cases} \partial_t \psi_n(t,x) = \partial_{xx}\psi_n(t,x) - \frac{1}{2}\psi_n^2(t,x) \\ \psi_n(0,x) = n\theta(x) \end{cases} \tag{5.12}$$

with the same boundary condition that (5.2) *is considered with, and $0 \leq \theta(x) \leq 1$ is a continuous function that satisfies the same boundary condition. Then $\{\psi_n\}$ is an increasing sequence of functions bounded above by function $\frac{2}{t}$.*

Proof. First we will prove that $\{\psi_n\}$ is an increasing sequence of functions.

Apply Lemma 5.1.2 to ψ_n which satisfies (5.12) with anyone of the three possible boundary conditions to see that $\psi_n(t,x) \geq 0$, for $(t,x) \in [0,T] \times [0,L]$. Suppose that $n_1 \leq n_2$, let $V(t,x) = \psi_{n_2}(t,x) - \psi_{n_1}(t,x)$, then we have

$$\begin{aligned}\partial_t V(t,x) &= \partial_t \left(\psi_{n_2}(t,x) - \psi_{n_1}(t,x)\right) \\ &= \partial_{xx} V(t,x) - \frac{1}{2}\left(\psi_{n_2}(t,x) + \psi_{n_1}(t,x)\right) V(t,x)\end{aligned}$$

Then, taking $c(t,x) = \frac{1}{2}\left(\psi_{n_2}(t,x) + \psi_{n_1}(t,x)\right)$, we have that when all ψ_n are considered with Neumann boundary conditions, V satisfies

$$\begin{cases} \partial_t V(t,x) - \partial_{xx} V(t,x) + c(t,x) V(t,x) = 0, & (t,x) \in (0,T) \times (0,L)\,, \\ V(0,x) = (n_2 - n_1)\,\theta(x) \geq 0, & x \in [0,L]\,, \\ \partial_x V(t,0) = \partial_x V(t,L) = 0, & t \in [0,T]\,, \end{cases} \tag{5.13}$$

when all ψ_n are considered with Dirichlet boundary conditions, the boundary condition in (5.13) becomes $V(t,0) = V(t,L) = 0$, for all $t \in [0,T]$, and when all ψ_n are considered with periodic boundary conditions, the boundary condition in (5.13) becomes $V(t,0) = V(t,L) = 0$, $\partial_x V(t,0) = \partial_x V(t,L) = 0$, for all $t \in [0,T]$. In all three cases we could use Lemma 5.1.2 to see that $\psi_{n_1}(t,x) \leq \psi_{n_2}(t,x)$ for $(t,x) \in [0,T] \times [0,L]$.

We are going to use the comparison lemma again to prove that $\{\psi_n\}$ is bounded. Let $u_n(t,x) = \frac{2n}{2+nt}$ for $(t,x) \in [0,T] \times [0,L]$. Then u_n satisfies

$$\begin{cases} \partial_t u_n(t,x) = \partial_{xx} u_n(t,x) - \frac{1}{2} u_n^2(t,x), \\ u_n(0,x) = n. \end{cases}$$

At the boundary, $\partial_x u_n(t,0) = \partial_x u_n(t,L) = 0$ and $u_n(t,0) = u_n(t,L) > 0$. Let $W(t,x) = u_n(t,x) - \psi_n(t,x)$, taking $\hat{c}(t,x) = \frac{1}{2}\left(u_n(t,x) + \psi_n(t,x)\right)$, we could see that $W(t,x)$ satisfies

$$\begin{cases} \partial_t W(t,x) - \partial_{xx} W(t,x) + \hat{c}(t,x) W(t,x) = 0 & (t,x) \in (0,T) \times (0,L)\,, \\ W(0,x) = n - n\theta(x) \geq 0 & x \in [0,L]\,, \end{cases}$$

It is not difficult to see that in any of the three different boundary conditions, we could use Lemma 5.1.2 to conclude that for any n, $\psi_n(t,x) \leq u_n(t,x) = \frac{2n}{2+nt} \leq \frac{2}{t}$, for all $(t,x) \in [0,T] \times [0,L]$. $\square$

Remark 5.1.4. As a consequence of this lemma, we could see that for any $(t,x) \in (0,T] \times [0,L]$, $\{\psi_n\}$ is a bounded increasing sequence which will converge. i.e. For $(t,x) \in (0,T] \times [0,L]$, $\psi(t,x) = \lim_{n\to\infty} \psi_n(t,x)$ is well defined.

Remark 5.1.5. If (5.1) is considered with Neumann boundary conditions, then each ψ_n should be considered with Neumann boundary conditions as we discussed before. Then we could replace the initial condition in (5.12) by $\psi_n(0,x) = n$. Since this would still make sure the limiting function will have infinite initial condition, and this does agree with the Neumann boundary condition. This is the reason why we introduced $\theta(x)$ in the first place, to make sure the initial condition agrees with Dirichlet boundary condition. Then we can see that $\frac{2n}{2+nt}$ is the solution to (5.12) with $\psi_n(0,x) = n$ as initial condition. Then, the limiting function is $\psi(t,x) = \lim_{n\to\infty} \psi_n(t,x) = \frac{2}{t}$. Note that it does satisfy the equation $\partial_t \psi(t,x) = \partial_{xx}\psi(t,x) - \frac{1}{2}\psi^2(t,x)$.

Lemma 5.1.6. *Suppose that $\{\phi_n\}$ is a sequence of functions that each of them satisfies the equation*

$$\begin{cases} \partial_t \phi_n(t,x) = \partial_{xx}\phi_n(t,x) - \psi_n(T-t,x)\phi_n(t,x), & (t,x) \in (0,T) \times (0,L), \\ \phi_n(0,x) = \zeta(x), & x \in [0,L], \end{cases} \tag{5.14}$$

with the same boundary condition that (5.2) *is considered with, and each ψ_n satisfies equation* (5.12) *with same boundary conditions. Then $\{\phi_n\}$ is a decreasing sequence of nonnegative functions.*

Proof. We will use the same method as we used twice in the proof of Lemma 5.1.3. First, it is easy to see that for all n, $\phi_n(t,x) \geq 0$, for all $(t,x) \in [0,T] \times [0,L]$. For any $n_1 \leq n_2$, let $V(t,x) = \phi_{n1}(t,x) - \phi_{n_2}(t,x)$, then V satisfies

$$\partial_t V(t,x) = \partial_{xx} V(t,x) - \psi_{n_1}(T-t,x)\,\phi_{n_1}(t,x) + \psi_{n_2}(T-t,x)\phi_{n_2}(t,x).$$

i.e. we have

$$\partial_t V(t,x) - \partial_{xx} V(t,x) + \psi_{n_1}(T-t,x)V(t,x) = \phi_{n_2}(t,x)\left(\psi_{n_2}(T-t,x) - \psi_{n_1}(T-t,x)\right).$$

Then $V(t,x)$ satisfies

$$\begin{cases} \partial_t V(t,x) - \partial_{xx} V(t,x) + \psi_{n_1}(T-t,x)V(t,x) \geq 0, & (t,x) \in (0,T) \times (0,L)\,, \\ V(0,x) = 0 & x \in [0,L] \end{cases}$$

It is easy to see that in any of the three possible boundary conditions we could use Lemma 5.1.2 to conclude that for any $(t,x) \in [0,T] \times [0,L]$

$$\phi_{n_1}(t,x) \geq \phi_{n_2}(t,x).$$

$\square$

Remark 5.1.7. As a consequence of this lemma, we could see that for any $(t,x) \in [0,T] \times [0,L]$, $\{\phi_n\}$ is a bounded decreasing sequence which will converge. i.e. For $(t,x) \in [0,T] \times [0,L]$, $\phi(t,x) = \lim_{n\to\infty} \phi_n(t,x)$ is well defined.

Remark 5.1.8. When (5.1) is considered with Neumann boundary conditions, then so are (5.14) and (5.12). As we discussed before in Remark 5.1.5, we could consider (5.12) with initial condition $\psi_n(0,x) = n$. Then we could use Feyman-Kac formula to prove that $\phi(T,x) = \lim_{n\to\infty} \phi_n(T,x) \equiv 0$ by the following argument. By Feyman-Kac, $\phi_n(t,x)$ can be written as

$$\phi_n(t,x) = E^x \left[\zeta(\sqrt{2}B_t) \exp\left\{ -\int_0^t \psi_n(T-t+s, \sqrt{2}B_s) ds \right\} \right]$$

where B is a doubly reflected Brownian motion with state space $[0,L]$ starting from $x \in [0,L]$. Then, by monotone convergence theorem and the fact that $\psi_n(t,x) = \frac{2n}{2+nt}$, we could see that

$$\begin{aligned} \phi(T,x) &= \lim_{n\to\infty} \phi_n(T,x) \\ &= \lim_{n\to\infty} E^x \left[\zeta(\sqrt{2}B_T) \exp\left\{ -\int_0^T \psi_n(s, \sqrt{2}B_s) ds \right\} \right] \\ &= \lim_{n\to\infty} E^x \left[\zeta(\sqrt{2}B_T) \exp\left\{ -\int_0^T \frac{2n}{2+2ns} ds \right\} \right] \\ &= 0. \end{aligned}$$

Therefore, $\phi(T,x) \equiv 0$.

Remark 5.1.9. As a special case, when (5.1) is considered with Neumann boundary conditions, as discussed above, by Feyman-Kac, $\phi_n(t,x)$ can be written as

$$\begin{aligned}
\phi_n(t,x) &= E\left[\zeta(x+\sqrt{2}B_t)\exp\left\{-\int_0^t \psi_n(T-t+s,x+\sqrt{2}B_s)ds\right\}\right] \\
&= E\left[\zeta(x+\sqrt{2}B_t)\right]\exp\left(-\int_0^t \frac{2n}{2+n\left(T-t+s\right)}ds\right) \\
&= E\left[\zeta(x+\sqrt{2}B_t)\right]\frac{\left(n\left(T-t\right)+2\right)^2}{\left(nT+2\right)^2}.
\end{aligned}$$

Then

$$\begin{aligned}
\phi(t,x) &= \lim_{n\to\infty}\phi_n(t,x) = E\left[\zeta(x+\sqrt{2}B_t)\right]\frac{\left(T-t\right)^2}{T^2} \\
&= E\left[\zeta(x+\sqrt{2}B_t)\right]\exp\left(-\int_0^t \frac{2}{T-t+s}ds\right) \\
&= E\left[\zeta(x+\sqrt{2}B_t)\exp\left\{-\int_0^t \psi(T-t+s,x+\sqrt{2}B_s)ds\right\}\right]
\end{aligned}$$

where $\psi(t,x)$ is as discussed in Remark 5.1.5. Then we could see that ϕ satisfies the equation $\partial_t\phi(t,x) = \partial_{xx}\phi(t,x) - \psi(T-t,x)\phi(t,x)$.

Remark 5.1.10. As we discussed in Remark 5.1.4 and 5.1.7, the function $\psi(t,x)$ and $\phi(t,x)$ are well defined. Although we did not manage to prove that they would satisfy the pair of PDEs

$$\begin{cases} \partial_t\psi(t,x) = \partial_{xx}\psi(t,x) - \frac{1}{2}\psi^2(t,x), \\ \partial_t\phi(t,x) = \partial_{xx}\phi(t,x) - \psi(T-t,x)\phi(t,x). \end{cases} \tag{5.15}$$

where they both satisfies the same boundary condition that (5.1) is considered with and $\phi(0,x) = \zeta(x)$, $\phi(T,x) \equiv 0$, we did show this is true when (5.1) is considered with Neumann boundary conditions. So we assume this is true in all three different boundary conditions. Then we would like to see what we will get if we put this ϕ in the rate function which we get from large deviations result. Since the rate function is defined in (5.5) and the assumption we just made

$$F(\phi) = \int_0^T\int_0^L \frac{(\phi_t - \phi_{xx})^2}{\phi}dxdt = \int_0^T\int_0^L \psi^2(T-t,x)\phi(t,x)dxdt.$$

Then again by the assumption,

$$\begin{aligned}
& \partial_t \left[\psi\left(T-t,x\right)\phi(t,x)\right] \\
= \;& \left[\partial_{xx}\phi(t,x)\right]\psi(T-t,x) - \left[\partial_{xx}\psi(T-t,x)\right]\phi(t,x) - \frac{1}{2}\psi^2(T-t,x)\phi(t,x).
\end{aligned}$$

Then integrate both sides over $[0,T]\times[0,L]$. By integration by parts formula and monotone convergence theorem, under each of the three possible boundary conditions, we could see that

$$\begin{aligned}
\frac{1}{2}F(\phi) &= \frac{1}{2}\int_0^T\int_0^L \psi^2(T-t,x)\phi(t,x)dxdt \\
&= \int_0^L \left[\psi(T,x)\phi(0,x) - \psi(0,x)\phi(T,x)\right]dx \\
&= \int_0^L \psi(T,x)\phi(0,x)dx \\
&= \lim_{n\to\infty}\int_0^L \psi_n(T,x)\phi(0,x)dx.
\end{aligned}$$

5.1.2 Standard martingale method to calculate $P\left\{u^\epsilon(T,x)=0,\forall x\in[0,L]\right\}$ for Super-Brownian motion

In this section, we will use a martingale method to calculate $P\left\{u^\epsilon(T,x)=0,\forall x\in[0,L]\right\}$, where u^ϵ satisfies (5.1) with one of the possible three boundary conditions. So that if it can be shown that the pair of functions $\phi(t,x)$ and $\psi(t,x)$ does satisfies the pair of PDEs (5.15) considered with the same boundary conditions and $\phi(T,x)\equiv 0$, as we pointed out in some special case, we can confirm that our candidate for the minimizer does give the right estimate on $P\left\{u^\epsilon(T,x)=0,\forall x\in[0,L]\right\}$. All these are shown in the following lemma.

Lemma 5.1.11. *Suppose that $u^\epsilon(t,x)$ satisfies* (5.1) *with one of the possible three boundary conditions. Suppose that $\psi_n(t,x)$ is the solution to* (5.12) *considered with the same boundary conditions. Then*

$$P\left\{u^\epsilon(T,x)=0,\forall x\in[0,L]\right\} = \lim_{n\to\infty}\exp\left[-\frac{1}{\epsilon^2}\left(\int_0^L u^\epsilon(0,x)\psi_n(T,x)dx\right)\right].$$

Proof. For any $h(t,x)\in C^{1,2}\left([0,T]\times[0,L]\right)$, satisfying the same boundary condition as (5.1) is considered with, by the definition of solution to an stochastic partial differential equation, we know that

$$\begin{aligned}
&\int_0^L \left[u^\epsilon(t,x)h(t,x) - u^\epsilon(0,x)h(0,x)\right] dx \\
=& \int_0^t \int_0^L u^\epsilon(s,x) \left(\partial_t h(s,x) + \partial_{xx} h(s,x)\right) dxds \\
&+ \int_0^t \int_0^L \epsilon\sqrt{u^\epsilon(s,x)} h(s,x) W(dxds).
\end{aligned}$$

Let ψ_n be the solution to (5.12), then it satisfies the same boundary conditions that (5.1) is considered with and by standard PDE theory it is in $C^{1,2}\left([0,T]\times[0,L]\right)$. Therefore, if we define

$$Z_t^n = \int_0^L \frac{1}{\epsilon^2} u^\epsilon(t,x)\psi_n(T-t,x)dx,$$

we could see that

$$\begin{aligned}
Z_t^n =& \int_0^L \frac{1}{\epsilon^2} u^\epsilon(0,x)\psi_n(T,x)dx + \int_0^t \int_0^L \frac{1}{\epsilon}\sqrt{u^\epsilon(s,x)}\psi_n(T-s,x)W(dxds) \\
&+ \int_0^t \int_0^L \frac{1}{\epsilon^2} u^\epsilon(s,x)\left[\partial_t\psi_n(T-s,x) + \partial_{xx}\psi_n(T-s,x)\right] dxds \\
=& \int_0^L \frac{1}{\epsilon^2} u^\epsilon(0,x)\psi_n(T,x)dx + \int_0^t \int_0^L \frac{1}{\epsilon}\sqrt{u^\epsilon(s,x)}\psi_n(T-s,x)W(dxds) \\
&+ \int_0^t \int_0^L \frac{1}{\epsilon^2} u^\epsilon(s,x)\frac{1}{2}\psi_n^2(T-s,x)dxds.
\end{aligned}$$

Let

$$N_t^n = -\int_0^t \int_0^L \frac{1}{\epsilon}\sqrt{u^\epsilon(s,x)}\psi_n(T-s,x)W(dxds),$$

then by the method we used in section 4.2 and 4.3, we could check that $\exp\left(N_t^n - \frac{1}{2}\left[N^n\right]_t\right)$ is a martingale. Meanwhile, since

$$\exp\left(-Z_t^n\right) = \exp\left(-\int_0^L \frac{1}{\epsilon^2} u^\epsilon(0,x)\psi_n(T,x)dx\right) \exp\left(N_t^n - \frac{1}{2}\left[N^n\right]_t\right),$$

we have $E\left[\exp\left(-Z_0\right)\right] = E\left[\exp\left(-Z_T\right)\right]$, i.e.

$$\begin{aligned} E\left[\exp\left(-\frac{1}{\epsilon^2}\int_0^L u^\epsilon(T,x)\psi_n(0,x)dx\right)\right] &= E\left[\exp\left(-\frac{1}{\epsilon^2}\int_0^L u^\epsilon(0,x)\psi_n(T,x)dx\right)\right] \\ &= \exp\left(-\frac{1}{\epsilon^2}\int_0^L u^\epsilon(0,x)\psi_n(T,x)dx\right) \end{aligned}$$

Also, we know that

$$\begin{aligned} & E\left[\exp\left(-\frac{1}{\epsilon^2}\int_0^L u^\epsilon(T,x)\psi_n(0,x)dx\right)\right] \\ = \ & E\left[\mathbf{1}_{\{u^\epsilon(T,x)=0,\forall x\in[0,L]\}}\exp\left(-\frac{1}{\epsilon^2}\int_0^L u^\epsilon(T,x)\psi_n(0,x)dx\right)\right] \\ & +E\left[\mathbf{1}_{\{\exists x\in[0,L],u^\epsilon(T,x)\neq 0\}}\exp\left(-\frac{1}{\epsilon^2}\int_0^L u^\epsilon(T,x)\psi_n(0,x)dx\right)\right] \\ = \ & P\left\{u^\epsilon(T,x)=0,\forall x\in[0,L]\right\} + E\left[\mathbf{1}_{\{\exists x\in[0,L],u^\epsilon(T,x)\neq 0\}}\exp\left(-\frac{1}{\epsilon^2}\int_0^L u^\epsilon(T,x)\psi_n(0,x)dx\right)\right]. \end{aligned}$$

Since $u^\epsilon(t,x)\geq 0$ a.s. and $\psi_n(0,x)=n\theta(x)$, where $\{x:\theta(x)=0\}$ is a null set of $[0,L]$, if $\mathbf{1}_{\{\exists x\in[0,L],u^\epsilon(T,x)\neq 0\}}=1$, we have $\int_0^L u(T,x)\theta(x)>0$, and hence by dominated convergence theorem, we have

$$\begin{aligned} & \lim_{n\to\infty} E\left[\mathbf{1}_{\{\exists x\in[0,L],u^\epsilon(T,x)\neq 0\}}\exp\left(-\frac{1}{\epsilon^2}\int_0^L u^\epsilon(T,x)\psi_n(0,x)dx\right)\right] \\ = \ & \lim_{n\to\infty} E\left[\mathbf{1}_{\{\exists x\in[0,L],u^\epsilon(T,x)\neq 0\}}\exp\left(-\frac{n}{\epsilon^2}\int_0^L u^\epsilon(T,x)\theta(x)dx\right)\right] \\ = \ & 0. \end{aligned}$$

Therefore we have proved that

$$\begin{aligned} & P\left\{u^\epsilon(T,x)=0,\forall x\in[0,L]\right\} \\ = \ & \lim_{n\to\infty} E\left[\exp\left(-\frac{1}{\epsilon^2}\int_0^L u^\epsilon(T,x)\psi_n(0,x)dx\right)\right] \\ = \ & \lim_{n\to\infty}\exp\left(-\frac{1}{\epsilon^2}\int_0^L u^\epsilon(0,x)\psi_n(T,x)dx\right). \end{aligned}$$

$\square$

Remark 5.1.12. The result from the above lemma agrees with the result by putting our candidate for the solution to the Euler-Lagrange equation in the rate function of large deviations. Note the large deviation only gives the exponential rate and an upper bound. The above shows that the upper bound is correct.

5.2 Calculation of $P\left(u^\epsilon(t,\cdot)\in A|u^\epsilon(T,x)=0,\forall x\in[0,L]\right)$ in the Super-Brownian motion case

After showing the ways to calculate $P\left(u^\epsilon(T,x)=0,\forall x\in[0,L]\right)$, it would be interesting to see whether we could calculate for any $t\in(0,T)$, the conditional probability $P\left(u^\epsilon(t,\cdot)\in A|u^\epsilon(T,x)=0,\forall x\in[0,L]\right)$, where A is a set of continuous functions on $[0,L]$. In the first half of this section, we will define a new probability measure Q, which is equivalent to P, by $Q\left(u^\epsilon(t,\cdot)\in A\right)=P\left(u^\epsilon(t,\cdot)\in A|u^\epsilon(T,x)=0,\forall x\in[0,L]\right)$. Then we will use Girsanov theorem to derive the SPDE that u^ϵ satisfies under the new measure Q. After we get that SPDE, we will prove a central limit type theorem, which will be the content of the second half of this section.

5.2.1 Girsanov change of measure

Lemma 5.2.1. *Suppose that* $u^\epsilon(t,x)$ *satisfies* (5.1) *on a probability space* $(\Omega,\mathscr{F},\mathscr{F}_t,P)$. *Define* Q *by*

$$Q\left(\Omega_0\right)=P\left(\Omega_0|u^\epsilon(T,x)=0,\forall x\in[0,L]\right),\ for\ \Omega_0\in\mathscr{F}.$$

Then, there exists a Q *space-time white noise process* $\widetilde{W}$ *based on* $[0,T]\times[0,L]$ *and under* Q, u^ϵ *satisfies*

$$\partial_t u^\epsilon(t,x)=\partial_{xx}u^\epsilon(t,x)-u^\epsilon(t,x)\psi(T-t,x)+\epsilon\sqrt{u^\epsilon(t,x)}\ddot{\widetilde{W}}_{tx},\qquad(5.16)$$

when the equation is considered on $[0,T)\times[0,L]$ *and* u^ϵ *satisfies the same boundary and initial condition as in* (5.1). *Here* $\psi(t,x)$ *is the function defined in Remark* 5.1.4.

Proof. Recall that the sequence $\{\psi_n\}$ is defined in (5.12). Let

$$M_t^n = \exp\left\{-\int_0^L \frac{1}{\epsilon^2} u^\epsilon(t,x)\psi_n(T-t,x)dx\right\}.$$

As shown in Lemma 5.1.11, M_t^n is an $\mathscr{F}_t$-martingale for any n. i.e. we have that a.s.

$$\begin{aligned} M_t^n &= E\left[M_T^n|\mathscr{F}_t\right] \\ &= E\left[\mathbf{1}_{\{u^\epsilon(T,x)=0,\forall x\in[0,L]\}} M_T^n|\mathscr{F}_t\right] + E\left[\mathbf{1}_{\{\exists x\in[0,L],u^\epsilon(T,x)\neq 0\}} M_T^n|\mathscr{F}_t\right]. \end{aligned}$$

Then using the same reasoning as used in Lemma 5.1.11, we can see that

$$\lim_{n\to\infty} M_t^n = E\left[\mathbf{1}_{\{u^\epsilon(T,x)=0,\forall x\in[0,L]\}}|\mathscr{F}_t\right] \quad a.s.$$

Then by the fact that $\{\psi_n\}$ is an increasing sequence and monotone convergence theorem we could see that for any $t\in[0,T]$,

$$M_t := \lim_{n\to\infty} M_t^n = \exp\left\{-\int_0^L \frac{1}{\epsilon^2} u^\epsilon(t,x)\psi(T-t,x)dx\right\}.$$

Also, by monotone convergence theorem for conditional expectations, we could see that M_t defined above is a martingale itself. i.e. we have shown that

$$E\left[\mathbf{1}_{\{u^\epsilon(T,x)=0,\forall x\in[0,L]\}}|\mathscr{F}_t\right] = M_t.$$

and M_t is an $\mathscr{F}_t$ martingale.

Then if we define a new probability measure Q by

$$\frac{dQ}{dP} = \frac{\mathbf{1}_{\{u^\epsilon(T,x)=0,\forall x\in[0,L]\}}}{P\{u^\epsilon(T,x)=0,\forall x\in[0,L]\}},\ on\left\{\mathscr{F}_T\right\},$$

Then,

$$Q\left(u^\epsilon(t,\cdot)\in A\right) = P\left(u^\epsilon(t,\cdot)\in A|u^\epsilon(T,x)=0,\forall x\in[0,L]\right),\ for\ all\ t\in[0,T].$$

Also we have that,

$$\frac{dQ}{dP}|_{\mathscr{F}_t} = E^P\left[\frac{dQ}{dP}|\mathscr{F}_t\right] = \frac{E^P\left[\mathbf{1}_{\{u^\epsilon(T,x)=0,\forall x\in[0,L]\}}|\mathscr{F}_t\right]}{P\{u^\epsilon(T,x)=0,\forall x\in[0,L]\}} = \frac{M_t}{P\{u^\epsilon(T,x)=0,\forall x\in[0,L]\}}.$$

On the other hand, using the result from Lemma 5.1.11, we could see that for any $t \in [0, T)$, for all n, we have

$$\begin{aligned}
& \frac{1}{P\left(u^\epsilon(T,x)=0, \forall x \in [0,L]\right)} M_t^n \\
= \quad & \exp\left\{-\int_0^t \int_0^L \frac{1}{2\epsilon^2} u^\epsilon(s,x)\psi_n^2(T-s,x)dxds - \int_0^t \int_0^L \frac{1}{\epsilon}\sqrt{u^\epsilon(s,x)}\psi_n(T-s,x)W(dxds)\right\} \\
& \cdot \exp\left\{-\int_0^L \frac{1}{\epsilon^2} u^\epsilon(0,x)\psi_n(T,x)dx + \int_0^L \frac{1}{\epsilon^2} u^\epsilon(0,x)\psi(T,x)dx\right\} \\
:= \quad & \exp\left\{N_t^n - \frac{1}{2}\left[N^n\right]_t + H_n\right\},
\end{aligned}$$

where

$$N_t^n = -\int_0^t \int_0^L \frac{1}{\epsilon}\sqrt{u^\epsilon(s,x)}\psi_n(T-s,x)W(dxds),$$

and

$$H_n = -\int_0^L \frac{1}{\epsilon^2} u^\epsilon(0,x)\psi_n(T,x)dx + \int_0^L \frac{1}{\epsilon^2} u^\epsilon(0,x)\psi(T,x)dx.$$

Using monotone convergence theorem again, we could see that for any $t \in [0, T)$, almost surely,

$$\begin{aligned}
\lim_{n\to\infty}\left[N^n\right]_t &= \int_0^t \int_0^L \frac{1}{\epsilon^2} u^\epsilon(s,x)\psi^2(T-s,x)dxds \\
&:= [N]_t\,,
\end{aligned}$$

where $N_t = -\int_0^t \int_0^L \frac{1}{\epsilon}\sqrt{u^\epsilon(s,x)}\psi(T-s,x)W(dxds)$. Also by monotone convergence theorem, we have $\lim_{n\to\infty} H_n = 0$. Therefore, we could write $M_t = \exp\left\{N_t - \frac{1}{2}[N]_t\right\}$.

By Girsanov theorem, we could say that there exists a new process $\widetilde{W}$ based on $[0,T]\times[0,L]$, such that for any Borel subset B of $[0,T]\times[0,L]$,

$$\widetilde{W}(B) = W(B) + \int_0^T \int_0^L \mathbf{1}_{\{(s,x)\in B\}} \frac{1}{\epsilon}\sqrt{u^\epsilon(s,x)}\psi(T-s,x)dxds.$$

Also, this new process is a white noise under the probability measure Q, and under this measure, u^ϵ satisfies (5.16). $\square$

5.2.2 A central limit type theorem

Lemma 5.2.2. *Suppose that function $\psi(t,x)$ is defined as in Remark* 5.1.5 *and ϕ is defined as in Remark* 5.1.9. *Suppose further that u^ϵ satisfies the stochastic partial differential equation* (5.16) *with initial condition $u^\epsilon(0,x) = \zeta(x)$ and Neumann boundary conditions. Here W is a white space time noise on probability space $\{\Omega, \mathscr{F}, P\}$ which is based on $[0,T] \times [0,L]$. Then for almost all $(t,x) \in [0,T) \times [0,L]$, we have*

$$\lim_{\epsilon \to 0} u^\epsilon(t,x) = \phi(t,x) \quad \textit{in } L^2.$$

Proof. We will first prove that for almost all $(t,x) \in [0,T) \times [0,L]$,

$$E\left[u^\epsilon(t,x)\right] = \phi(t,x). \tag{5.17}$$

Then we will prove that for almost all $(t,x) \in [0,T) \times [0,L]$,

$$\lim_{\epsilon \to 0} E\left[\left(u^\epsilon(t,x) - \phi(t,x)\right)^2\right] = 0. \tag{5.18}$$

To prove (5.17), we note that for any $h(t,x) \in C^{1,2}\left([0,T] \times [0,L]\right)$ satisfying Neumann boundary conditions, by the definition of solution to stochastic partial differential equations, we know that

$$\begin{aligned} & \int_0^L \left[u^\epsilon(t,x)h(t,x) - u^\epsilon(0,x)h(0,x)\right] dx \qquad (5.19) \\ = & \int_0^t \int_0^L u^\epsilon(s,x)\left[\partial_{xx}h(s,x) - \psi(T-s,x)h(s,x) + \partial_t h(s,x)\right] dxds \\ & + \int_0^t \int_0^L \epsilon\sqrt{u^\epsilon(s,x)}h(s,x)W(dxds) \end{aligned}$$

Note that $\int_0^t \int_0^L \epsilon\sqrt{u^\epsilon(s,x)}h(s,x)W(dxds)$ is a martingale. Since

$$\sup_{t<T} E\left[\int_0^L u^\epsilon(t,x)dx\right] < \infty,$$

taking expectation of both sides of (5.19) will give us

$$\begin{aligned} & \int_0^L E\left[u^\epsilon(t,x)\right] h(t,x)dx - \int_0^L u^\epsilon(0,x)h(0,x)dx \qquad (5.20) \\ = & \int_0^t \int_0^L E\left[u^\epsilon(t,x)\right]\left[\partial_{xx}h(s,x) - \psi(T-s,x)h(s,x) + \partial_t h(s,x)\right] dxds, \end{aligned}$$

from which we can say that $E\left[u^\epsilon(t,x)\right]$ is a weak solution to the PDE that ϕ satisfies in Remark 5.1.9. Since the weak solution is unique, we can see that we have (5.17), for almost all $(t,x) \in [0,T) \times [0,L]$.

Next, we are going to prove (5.18). Let

$$M_1(t,x) = E\left[u^\epsilon(t,x)\right],$$

$$M_2(t,x,y) = E\left[u^\epsilon(t,x)u^\epsilon(t,y)\right].$$

As we have shown that $\phi(t,x) = M_1(t,x)$ for almost all $(t,x) \in [0,T) \times [0,L]$, we could see that

$$E\left[\left(u^\epsilon(t,x) - \phi(t,x)\right)^2\right] = M_2(t,x,x) - M_1^2(t,x).$$

Roughly speaking, our way to prove (5.18) is first to prove that $M_2(t,x,y)$ satisfies a PDE in a certain way, and then to evaluate $M_2(t,x,y) - M_1(t,x)M_1(t,y)$ along the line $x = y$.

For functions $h(t,x), \bar{h}(t,x) \in C^{1,2}\left([0,T] \times [0,L]\right)$, let

$$Z_t = \int_0^L u^\epsilon(t,x)h(t,x)dx,$$

$$\bar{Z}_t = \int_0^L u^\epsilon(t,x)\bar{h}(t,x)dx.$$

Then use Ito's formula and the representation (5.19), we could see that

$$\begin{aligned} & Z_t\bar{Z}_t - Z_0\bar{Z}_0 \qquad (5.21) \\ = & \int_0^t\int_0^L\int_0^L u^\epsilon(s,x)u^\epsilon(s,y)\Xi(s,x,y)dxdyds \\ & +\int_0^t\int_0^L u^\epsilon(s,x)h(s,x)dxd\bar{M}_s + \int_0^t\int_0^L u^\epsilon(s,x)\bar{h}(s,x)dxdM_s \\ & +\epsilon^2\int_0^t\int_0^L u^\epsilon(s,x)h(s,x)\bar{h}(s,x)dxds, \end{aligned}$$

where

$$\begin{aligned} \Xi(s,x,y) &= \triangle_{xy}\left(h(s,x)\bar{h}(s,y)\right) - \left(\psi(T-s,x)+\psi(T-s,y)\right)\left(h(s,x)\bar{h}(s,y)\right) \\ & \quad +\partial_s\left(h(s,x)\bar{h}(s,y)\right), \\ \bar{M}_t &= \epsilon\int_0^t\int_0^L \sqrt{u^\epsilon(s,x)}\bar{h}(s,x)W(dxds), \\ M_t &= \epsilon\int_0^t\int_0^L \sqrt{u^\epsilon(s,x)}h(s,x)W(dxds). \end{aligned}$$

Take expectation of both sides of (5.21) will give us

$$\begin{aligned} & \int_0^L\int_0^L M_2(t,x,y)h(t,x)\bar{h}(t,y)dxdy \qquad (5.22) \\ = & \int_0^L\int_0^L M_2(0,x,y)h(0,x)\bar{h}(0,y)dxdy + \int_0^t\int_0^L\int_0^L M_2(s,x,y)\Xi(s,x,y)dxdyds \\ & +\epsilon^2\int_0^t\int_0^L M_1(s,x)h(s,x)\bar{h}(s,x)dxds, \end{aligned}$$

Since in (5.22) the terms involve $h(t,x)\bar{h}(t,y)$ are all linear in it, we could see that if we replace $h(t,x)\bar{h}(t,y)$ in the left hand side of (5.22) by $\sum_{i=1}^n h_i(t,x)\bar{h}_i(t,y)$, where for each i, $h_i(t,x), \bar{h}_i(t,x) \in C^{1,2}([0,T]\times[0,L])$, we will get

$$\begin{aligned}
& \int_0^L \int_0^L M_2(t,x,y)\left[\sum_{i=1}^n h_i(t,x)\bar{h}_i(t,y)\right] dxdy \qquad (5.23)\\
= & \int_0^L \int_0^L M_2(0,x,y)\left[\sum_{i=1}^n h_i(0,x)\bar{h}_i(0,y)\right] dxdy + \int_0^t \int_0^L \int_0^L M_2(s,x,y)\Xi^n(s,x,y)dxdyds \\
& +\epsilon^2 \int_0^t \int_0^L M_1(s,x)\left[\sum_{i=1}^n h_i(s,x)\bar{h}_i(s,x)\right])dxds,
\end{aligned}$$

where

$$\begin{aligned}
\Xi^n(s,x,y) &= \triangle_{xy}\left[\sum_{i=1}^n h_i(s,x)\bar{h}_i(s,y)\right] \\
&\quad -(\psi(T-s,x)+\psi(T-s,y))\left[\sum_{i=1}^n h_i(s,x)\bar{h}_i(s,y)\right] \\
&\quad +\partial_s\left[\sum_{i=1}^n h_i(s,x)\bar{h}_i(s,y)\right]
\end{aligned}$$

Next we will try to use Green's function for the two dimensional heat equation to represent $M_2(t,x,y)$. Let G be the Green's function for the two dimensional Laplacian operator with Neumann boundary conditions. Let W_i be the eigenfunction of the one dimensional Laplacian with the Neumann boundary condition corresponding with eigenvalue λ_i. i.e. $-\triangle W_i = \lambda_i W_i$. For $t \in (0,T)$ fixed, for all $0 \le s \le t$, define

$$G^n_{t-s}(\xi,\eta;x,y) = \sum_{i,j=1}^n \exp\left(-(\lambda_i+\lambda_j)(t-s)\right) W_i(x)W_j(y)W_i(\xi)W_j(\eta).$$

Then for any $(\xi,\eta) \in [0,L]\times[0,L]$ fixed, for any $0 \le s < t$, we have

$$\lim_{n\to\infty} \int_0^L \int_0^L \left(G_{t-s}(\xi,\eta;x,y) - G^n_{t-s}(\xi,\eta;x,y)\right)^2 dxdy = 0. \qquad (5.24)$$

Meanwhile, by the definition of $G^n_{t-s}(\xi,\eta;x,y)$, in (5.23), we could take

$$\sum_{i=1}^{n} h_i(s,x)\bar{h}_i(s,y) = G^n_{t-s}(\xi,\eta;x,y).$$

Then we have

$$\Xi^n(s,x,y) = -\left(\psi(T-s,x)+\psi(T-s,y)\right)G^n_{t-s}(\xi,\eta;x,y),$$

and (5.23) becomes

$$\begin{aligned}
& \int_0^L\int_0^L M_2(t,x,y)G^n_0(\xi,\eta;x,y)dxdy \qquad (5.25)\\
= & \int_0^L\int_0^L M_2(0,x,y)G^n_t(\xi,\eta;x,y)dxdy\\
& -\int_0^t\int_0^L\int_0^L M_2(s,x,y)G^n_{t-s}(\xi,\eta;x,y)\left(\psi(T-s,x)+\psi(T-s,y)\right)dxdy\\
& +\int_0^t\int_0^L\int_0^L M_1(s,x)G^n_{t-s}(\xi,\eta;x,x)dxds
\end{aligned}$$

Using Cauchy-Schwarz inequality, we could see that

$$\begin{aligned}
& \left|\int_0^L\int_0^L M_2(0,x,y)G_t(\xi,\eta;x,y)dxdy - \int_0^L\int_0^L M_2(0,x,y)G^n_t(\xi,\eta;x,y)dxdy\right|\\
\leq & \left\{\int_0^L\int_0^L M_2^2(0,x,y)dxdy\right\}^{1/2}\left\{\int_0^L\int_0^L \left(G_t(\xi,\eta;x,y)-G^n_t(\xi,\eta;x,y)\right)^2 dxdy\right\}^{1/2}
\end{aligned}$$

Therefore, for any fixed $(\xi,\eta)\in[0,L]\times[0,L]$

$$\lim_{n\to\infty}\int_0^L\int_0^L M_2(0,x,y)G^n_t(\xi,\eta;x,y)dxdy = \int_0^L\int_0^L M_2(0,x,y)G_t(\xi,\eta;x,y)dxdy \qquad (5.26)$$

Similarly, if we let

$$H_n(\xi,\eta) = \int_0^t\int_0^L\int_0^L M_2(s,x,y)\left(\psi(T-s,x)+\psi(T-s,y)\right)G^n_{t-s}(\xi,\eta;x,y)dxdyds$$

$$H(\xi,\eta) = \int_0^t\int_0^L\int_0^L M_2(s,x,y)\left(\psi(T-s,x)+\psi(T-s,y)\right)G_{t-s}(\xi,\eta;x,y)dxdyds$$

then using the upper bound for function ψ, we have

$$
\begin{aligned}
& |H_n(\xi,\eta) - H(\xi,\eta)| \\
\leq \; & \int_0^t \frac{4}{T-s} \left\{ \int_0^L \int_0^L M_2^2(s,x,y) dxdy \right\}^{1/2} \\
& \cdot \left\{ \int_0^L \int_0^L \left(G_{t-s}(\xi,\eta;x,y) - G_{t-s}^n(\xi,\eta;x,y) \right)^2 dxdy \right\}^{1/2} ds
\end{aligned}
$$

Since we know that $\sup_{0\leq s\leq t} \int_0^L \int_0^L \left[E\left(u^\epsilon(s,x) u^\epsilon(s,y) \right) \right]^2 dxdy \leq C < \infty$, where C is a constant, we can use the Cauchy-Schwarz inequality again to see that

$$
\begin{aligned}
& |H_n(\xi,\eta) - H(\xi,\eta)| \\
\leq \; & \sqrt{C} \left\{ \int_0^t \left(\frac{4}{T-s} \right)^2 ds \right\}^{1/2} \left\{ \int_0^t \int_0^L \int_0^L \left(G_{t-s}(\xi,\eta;x,y) - G_{t-s}^n(\xi,\eta;x,y) \right)^2 dxdyds \right\}^{1/2}
\end{aligned}
$$

It is easy to see that

$$
\begin{aligned}
& \int_0^t \int_0^L \int_0^L \left(G_{t-s}(\xi,\eta;x,y) - G_{t-s}^n(\xi,\eta;x,y) \right)^2 dxdyds \qquad (5.27) \\
= \; & \int_0^t \int_0^L \int_0^L \sum_{i,j=n+1}^{\infty} \exp\left(-2\left(\lambda_i + \lambda_j \right)(t-s) \right) W_i^2(\xi) W_j^2(\eta) W_i^2(x) W_j^2(y) dxdyds \\
= \; & \sum_{i,j=n+1}^{\infty} \frac{1 - e^{-2(\lambda_i+\lambda_j)t}}{2\left(\lambda_i + \lambda_j \right)} W_i^2(\xi) W_j^2(\eta)
\end{aligned}
$$

As $\sum_{i,j=1}^{\infty} \frac{1}{2(\lambda_i+\lambda_j)} < \infty$, we have that for any fixed $(\xi,\eta) \in [0,L] \times [0,L]$

$$
\lim_{n\to\infty} H_n(\xi,\eta) = H(\xi,\eta) \qquad (5.28)
$$

Next, if we let

$$
I_n(\xi,\eta) = \int_0^t \int_0^L M_1(s,x) G_{t-s}^n(\xi,\eta;x,x) dxds
$$

$$
I(\xi,\eta) = \int_0^t \int_0^L M_1(s,x) G_{t-s}(\xi,\eta;x,x) dxds
$$

We have that

$$
\begin{aligned}
& |I_n(\xi,\eta) - I(\xi,\eta)| \\
\leq & \int_0^t \left\{ \int_0^L M_1^2(s,x)dx \right\}^{1/2} \\
& \cdot \left\{ \int_0^L \left(G_{t-s}(\xi,\eta;x,x) - G_{t-s}^n(\xi,\eta;x,x) \right)^2 dx \right\}^{1/2} ds
\end{aligned}
$$

Since we know that $\sup_{0\leq s\leq t} \int_0^L \left[E\left(u^\epsilon(s,x)\right)\right]^2 dx < \infty$, and we can do the same calculation as we did in (5.24), we can conclude that for any fixed $(\xi,\eta) \in [0,L]\times[0,L]$

$$
\lim_{n\to\infty} I_n(\xi,\eta) = I(\xi,\eta) \tag{5.29}
$$

Next, since $\int_0^L \int_0^L \left[E\left(u^\epsilon(t,x)u^\epsilon(t,y)\right)\right]^2 dxdy < \infty$, and W_iW_j forms a orthonormal basis for $L^2\left([0,L]\times[0,L]\right)$, we have

$$
\begin{aligned}
& \lim_{n\to\infty} \int_0^L \int_0^L M_2(t,x,y) G_0^n(\xi,\eta;x,y)dxdy \qquad (5.30)\\
= & \lim_{n\to\infty} \int_0^L \int_0^L M_2(t,x,y) \sum_{i,j=1}^N W_i(\xi)W_j(\eta)W_i(x)W_j(y)dxdy \\
= & \lim_{n\to\infty} \sum_{i,j=1}^N \left(\int_0^L \int_0^L M_2(t,x,y)W_i(x)W_j(y)dxdy \right) W_i(\xi)W_j(\eta) \\
= & M_2(t,\xi,\eta)
\end{aligned}
$$

Finally, if we let $n\to\infty$ on both sides of (5.25), noting (5.26), (5.28), (5.29) and (5.30), we will have

$$
\begin{aligned}
M_2(t,\xi,\eta) = & \int_0^L \int_0^L M_2(0,x,y) G_t(\xi,\eta;x,y)dxdy \qquad (5.31)\\
& - \int_0^t \int_0^L \int_0^L M_2(s,x,y) \left(\psi(T-s,x) + \psi(T-s,y)\right) G_{t-s}(\xi,\eta;x,y)dxdyds \\
& + \epsilon^2 \int_0^t \int_0^L M_1(s,x) G_{t-s}(\xi,\eta;x,x)dxds
\end{aligned}
$$

Use the same method as we did in proving (5.31), we could easily see that

$$\begin{aligned} & M_1(t,\xi)M_1(t,\eta) \\ = & \int_0^L\int_0^L M_1(0,x)M_1(0,y)G_t(\xi,\eta;x,y)dxdy \\ & -\int_0^t\int_0^L\int_0^L M_1(s,x)M_1(s,y)\left(\psi(T-s,x)+\psi(T-s,y)\right)G_{t-s}(\xi,\eta;x,y)dxdyds \end{aligned} \tag{5.32}$$

If we let $R_t = \sup_{\xi,\eta}|M_2(t,\xi,\eta) - M_1(t,\xi)M_1(t,\eta)|$, from (5.31) and (5.32), we can see that

$$\begin{aligned} R_t & \leq \int_0^t\int_0^L\int_0^L R_s\left(\psi(T-s,x)+\psi(T-s,y)\right)G_{t-s}(\xi,\eta;x,y)dxdyds \\ & \quad +\epsilon^2\int_0^t\int_0^L\int_0^L M_1(s,x)G_{t-s}(\xi,\eta;x,y)\delta_x(y)dxdyds \\ & \leq \int_0^t R_s\frac{C_1}{T-s}ds + C_2\epsilon^2 t \end{aligned}$$

where C_1 and C_2 are constants. Then by Gronwall's inequality, we have that

$$R_t \leq \epsilon^2 C_2 T\exp(C_1\ln T),$$

and from this we could conclude that (5.18) is true.

□

Lemma 5.2.3. *Suppose that $u^\epsilon(t,x)$ satisfies (**??**), where $\psi(t,x)$ and $\phi(t,x)$ are same as in Lemma 5.2.2. Let*

$$V^\epsilon(t,x) = \frac{u^\epsilon(t,x)-\phi(t,x)}{\epsilon}.$$

Suppose further that $Z(t,x)$ satisfies

$$\begin{aligned} \partial_t Z(t,x) & = \partial_{xx}Z(t,x) - \psi(T-s,x)Z(t,x) + \sqrt{\phi(t,x)}\ddot{W}_{tx} \\ Z(0,x) & = 0 \end{aligned} \tag{5.33}$$

*where W is the same white space time noise appeared in (**??**), and Z satisfies Neumann boundary condition. Then for almost all fixed $(t,x) \in [0,T) \times$*

$[0, L]$,

$$\lim_{\epsilon \to 0} V^\epsilon(t, x) = Z(t, x) \ \ in\ L^2.$$

Proof. Since V^ϵ satisfies

$$\begin{aligned} \partial_t V^\epsilon(t, x) &= \partial_{xx} V^\epsilon(t, x) - \psi(T - s, x) V^\epsilon(t, x) + \sqrt{u^\epsilon(t, x)} \ddot{W}_{tx} \\ V^\epsilon(0, x) &= 0, \end{aligned}$$

if we define $Y(t, x) = Z(t, x) - V^\epsilon(t, x)$, then $Y(t, x)$ satisfies

$$\begin{aligned} \partial_t Y(t, x) &= \partial_{xx} Y(t, x) - \psi(T - s, x) Y(t, x) + \left(\sqrt{\phi(t, x)} - \sqrt{u^\epsilon(t, x)} \right) \ddot{W}_{tx} \\ Y(0, x) &= 0. \end{aligned}$$

Then we could use the definition of solution of an SPDE and the Ito's formula, the same method as we used to derive (5.21) to get that for any $h(t, x), \bar{h}(t, x) \in C^{1,2}([0, T] \times [0, L])$

$$\begin{aligned} & \int_0^L \int_0^L Y(t, x) Y(t, y) h(t, x) \bar{h}(t, y) dx dy \qquad (5.34) \\ = & \int_0^t \int_0^L \int_0^L Y(s, x) Y(s, y) \Xi(s, x, y) dx dy ds \\ & + \int_0^t \int_0^L Y(s, x) h(s, x) dx d\bar{M}_s + \int_0^t \int_0^L Y(s, x) \bar{h}(s, x) dx dM_s \\ & + \frac{1}{2} \int_0^t \int_0^L \int_0^L h(s, x) \bar{h}(s, y) \left(\sqrt{\phi(t, x)} - \sqrt{u^\epsilon(t, x)} \right)^2 \delta_x(y) dx dy ds, \end{aligned}$$

where $\Xi(s, x, y)$ is the same as in (5.21), and

$$\begin{aligned} \bar{M}_t &= \int_0^t \int_0^L \left(\sqrt{\phi(s, x)} - \sqrt{u^\epsilon(s, x)} \right) \bar{h}(s, x) W(dx ds), \\ M_t &= \int_0^t \int_0^L \left(\sqrt{\phi(s, x)} - \sqrt{u^\epsilon(s, x)} \right) h(s, x) W(dx ds). \end{aligned}$$

Since we know that

$$\sup_{t\leq T}\int_0^L\int_0^L E\left[Y(t,x)Y(t,y)\right]h(t,x)\bar{h}(t,y)dxdy<\infty,$$

$$E\left[\int_0^t\int_0^L Y(s,x)h(s,x)dxd\bar{M}_s\right]=0,$$

$$E\left[\int_0^t\int_0^L Y(s,x)\bar{h}(s,x)dxdM_s\right]=0,$$

we could take expectation over both sides of (5.34) and if we let $M(t,x,y)=E\left[Y(t,x)Y(t,y)\right]$, we will get

$$\begin{aligned}
&\int_0^L\int_0^L M(t,x,y)h(t,x)\bar{h}(t,y)dxdy \qquad (5.35)\\
=&\int_0^t\int_0^L\int_0^L M(s,x,y)\Xi(s,x,y)dxdyds\\
&+\frac{1}{2}\int_0^t\int_0^L\int_0^L h(s,x)\bar{h}(s,y)\left(\sqrt{\phi(t,x)}-\sqrt{u^\epsilon(t,x)}\right)^2\delta_x(y)dxdyds.
\end{aligned}$$

Then, since we know that

$$\sup_{t\leq T}\int_0^L\int_0^L\left[Y(t,x)Y(t,y)\right]^2dxdy<\infty,$$

we could use the Green's function approximation method we used in proving (5.31) to get

$$\begin{aligned}
&M(t,x,y) \qquad (5.36)\\
=&\int_0^t\int_0^L\int_0^L M(s,x,y)\left(\psi(T-s,\xi)+\psi(T-s,\eta)\right)G_{t-s}(x,y;\xi,\eta)d\xi d\eta ds\\
&+\frac{1}{2}\int_0^t\int_0^L\int_0^L G_{t-s}(x,y;\xi,\eta)\delta_\xi(\eta)E\left[\left(\sqrt{\phi(s,\xi)}-\sqrt{u^\epsilon(s,\xi)}\right)^2\right]d\xi d\eta ds
\end{aligned}$$

If we could show that for any $(s,\xi)\in(0,T)\times[0,L]$,

$$E\left[\left(\sqrt{\phi(s,\xi)}-\sqrt{u^\epsilon(s,\xi)}\right)^2\right]\leq\epsilon C \qquad (5.37)$$

for some constant C, then we could use Gronwall inequality to show that for any $t \in (0,T)$

$$\sup_{x,y\in[0,L]\times[0,L]} |E[Y(t,x)Y(t,y)]| \leq \epsilon C_T$$

where C_T is a positive constant depending on T, and hence for any $(t,x) \in (0,T) \times [0,L]$

$$E\left[(V^\epsilon(t,x) - Z(t,x))^2\right] \leq \epsilon C_T,$$

i.e. $\lim_{\epsilon\to 0} V^\epsilon(t,x) = Z(t,x)$ *in* L^2. So the rest of the proof will be used to show that (5.37) is true. As shown at the end of Lemma (5.2.2), we know that for any $(t,x) \in (0,T) \times [0,L]$,

$$E\left[(u^\epsilon(t,x) - \phi(t,x))^2\right] \leq \hat{C}\epsilon^2.$$

Then if $\phi(t,x) \leq \epsilon$, let set $A := \{\omega \in \Omega : u^\epsilon(t,x) \leq \epsilon\}$, we can see that

$$\begin{aligned}
E\left[\left(\sqrt{u^\epsilon} - \sqrt{\phi}\right)^2\right] &= E\left[\mathbf{1}_A\left(\sqrt{u^\epsilon} - \sqrt{\phi}\right)^2\right] + E\left[\mathbf{1}_{A^c}\left(\sqrt{u^\epsilon} - \sqrt{\phi}\right)^2\right] \\
&\leq E\left[\mathbf{1}_A 2\left(u^\epsilon + \phi\right)\right] + E\left[\mathbf{1}_{A^c}\frac{(u^\epsilon - \phi)^2}{(\sqrt{u^\epsilon} + \sqrt{\phi})^2}\right] \\
&\leq 4\epsilon + \frac{1}{\epsilon}E\left[(u^\epsilon - \phi)^2\right] \\
&\leq \left(4 + \hat{C}\right)\epsilon.
\end{aligned}$$

If $\phi(t,x) > \epsilon$, then

$$E\left[\left(\sqrt{u^\epsilon} - \sqrt{\phi}\right)^2\right] = E\left[\frac{(u^\epsilon - \phi)^2}{(\sqrt{u^\epsilon} + \sqrt{\phi})^2}\right] \leq \hat{C}\epsilon$$

Therefore (5.37) is true and hence is the conclusion of this lemma. $\square$

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